620-295 Real Analysis with applications

Lecture 22, 30 April 2010 Fourier series and 5(2)

Taylor series at $x=0$ are about expanding functions $f(x)$ in powers of x.

Example Find the Taylor series of $f(x) = \cos^2 x$ at $x=0$.

Since
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1-\cos^2 x)$$
$$= 2\cos^2 x - 1,$$

$$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$$
$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)$$
$$= \frac{3}{2} - \frac{2x^2}{2!} + 2^3\frac{x^4}{4!} - 2^5\frac{x^6}{4!} + \cdots$$

Taylor series at $x=3$ are about expanding functions $f(x)$ in powers of $x-3$.

Example Find the Taylor series of $f(x) = x^3$ at $x=3$.

$$x^3 = (x-3+3)^3 = (x-3)^3 + 3(x-3)^2 3 + 3(x-3)\cdot 3^2 + 3^3$$
$$= 27 + 27(x-3) + 9(x-3)^2 + (x-3)^3.$$

Fourier series are about expanding functions $f(x)$ in powers of e^{ix}

Since
$$\sin kx = \frac{e^{ikx}-e^{-ikx}}{2i} \quad \text{and} \quad \cos kx = \frac{e^{ix}+e^{-ix}}{2}$$

$$e^{ikx} = \cos kx + i\sin kx \quad \text{and} \quad e^{-ikx} = \cos kx - i\sin kx,$$

Fourier series are about expanding functions $f(x)$ in terms of $\cos kx$ and $\sin kx$.

<u>Proposition</u> Let $k \in \mathbb{Z}$

(a) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,dx = \begin{cases} 0, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0. \end{cases}$$

(b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}e^{-ilx}\,dx = \delta_{kl}, \quad \text{where } \delta_{kl} = \begin{cases} 0, & \text{if } k \neq l, \\ 1, & \text{if } k = l. \end{cases}$$

(c) If
$$f(x) = c_0 + c_1 e^{ix} + c_{-1}e^{-ix} + c_2 e^{2ix} + c_{-2}e^{-i2x} + \cdots$$
then
$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{i(-kx)}\,dx$$

<u>Proof</u> (a) <u>Case 1</u>: $k=0$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i0x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{1}{2\pi}\left(x\Big|_{x=-\pi}^{x=\pi}\right) = \frac{1}{2\pi}(\pi-(-\pi))$$
$$= \frac{1}{2\pi}\cdot 2\pi = 1.$$

Case 2: If $k \neq 0$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,dx = \frac{1}{2\pi}\left(\frac{e^{ikx}}{ik}\Big|_{x=-\pi}^{x=\pi}\right)$$

$$= \frac{1}{2\pi ik}\left(e^{i\pi k} - e^{-i\pi k}\right) = \frac{1}{2\pi ik}\left((-1)^k - (-1)^{-k}\right) = 0.$$

(b) To show: $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\,dx = \delta_{kl}$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)x}\,dx$$

$$= \begin{cases} 0, & \text{if } k-l \neq 0 \\ 1, & \text{if } k-l = 0 \end{cases} = \begin{cases} 0, & \text{if } k \neq l \\ 1, & \text{if } k = l \end{cases} = \delta_{kl}$$

(c) Assume $f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{i2x} + c_{-2} e^{-i2x} + \dots$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + \dots\right) e^{-ikx}\,dx$$

$$= c_0 \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\,dx + c_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix} e^{-ikx}\,dx + c_{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix} e^{-ikx}\,dx + \dots$$

$$= 0 + 0 + \dots + 0 + c_k \cdot 1 + 0 + 0 + \dots = c_k.$$

So

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx.$$

<u>Example</u> Consider $f:[-\pi,\pi]\to\mathbb{R}$ given by $f(x)=x^2$.
Find the expansion

$$f(x)=c_0+c_1e^{ix}+c_{-1}e^{-ix}+c_2e^{2ix}+c_{-2}e^{-2ix}+\dots$$

First find c_0:

$$c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{2\pi}\left(\frac{x^3}{3}\Big|_{x=-\pi}^{x=\pi}\right)$$

$$=\frac{1}{2\pi}\left(\frac{\pi^3}{3}-\frac{(-\pi)^3}{3}\right)=\frac{1}{2\pi}\,\frac{2\pi^3}{3}=\frac{\pi^2}{3}.$$

To find $c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2e^{-ikx}\,dx$, first do the integral

$$\int x^2e^{-ikx}\,dx=x^2\frac{e^{-ikx}}{-ik}-\int 2x\frac{e^{-ikx}}{-ik}\,dx \qquad \left(\begin{matrix}\text{by integration}\\ \text{by parts}\end{matrix}\right)$$

$$=\frac{ix^2}{k}e^{-ikx}-\left(2x\frac{e^{-ikx}}{(-ik)^2}-\int 2\frac{e^{-ikx}}{(-ik)^2}\,dx\right)$$

$$=\frac{ix^2}{k}e^{-ikx}+\frac{2x}{k^2}e^{-ikx}+2\frac{e^{-ikx}}{(-ik)^3}$$

So

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2e^{-ikx}=\frac{1}{2\pi}\left(\frac{ix^2}{k}e^{-ikx}+\frac{2x}{k^2}e^{-ikx}-\frac{2i}{k^3}e^{-ikx}\right)\Bigg|_{x=-\pi}^{x=\pi}$$

$$=\frac{1}{2\pi}\left(\frac{i\pi^2}{k}+\frac{2\pi}{k^2}-\frac{2i}{k^3}\right)e^{-i\pi k}-\frac{1}{2\pi}\left(\frac{i\pi^2}{k}-\frac{2\pi}{k^2}-\frac{2i}{k^3}\right)e^{i\pi k}$$

$$=\frac{1}{2\pi}\left(\frac{i\pi^2}{k}+\frac{2\pi}{k^2}-\frac{2i}{k^3}\right)(-1)^k-\frac{1}{2\pi}\left(\frac{i\pi^2}{k}-\frac{2\pi}{k^2}-\frac{2i}{k^3}\right)(-1)^k$$

So $$c_k = \frac{1}{2\pi}(-1)^k\left(\frac{2\pi}{k^2}+\frac{2\pi}{k^2}\right) = (-1)^k\frac{4\pi}{2\pi k^2} = (-1)^k\frac{2}{k^2}.$$

So

$$x^2 = c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k}e^{-ikx}$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty} (-1)^k\frac{2}{k^2}e^{ikx} + (-1)^{-k}\frac{2}{(-k)^2}e^{-ikx}$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k\frac{2}{k^2}\left(e^{ikx}+e^{-ikx}\right)$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k\frac{2}{k^2}\cdot 2\cos kx$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}4\cos kx$$

$$= \frac{\pi^2}{3} - 4\cos kx + \frac{1}{4}4\cos 2x - \frac{1}{9}4\cos 3x + \frac{1}{16}4\cos 4x + \cdots$$

If $x=\pi$ then

$$\pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}4\cos k\pi$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k\frac{4}{k^2}(-1)^k$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4}{k^2}.$$

So $$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4}\left(\pi^2-\frac{\pi^2}{3}\right) = \frac{1}{4}\,\frac{2\pi^2}{3} = \frac{\pi^2}{6}.$$