620-295 Real Analysis with applications Lect. 20, 23.04.2010 (1)

The fundamental theorem of calculus

Write

$$\int f(x)\,dx = A(x) + c \quad \text{if} \quad \frac{d}{dx}A(x) = f(x)$$

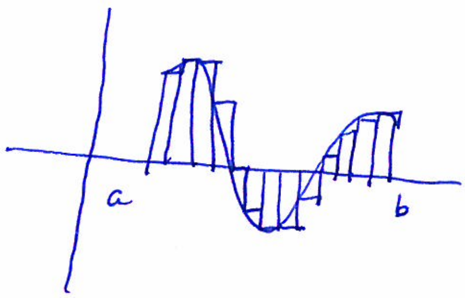
The fundamental theorem of calculus says

$$\begin{array}{l}\text{Area under } f(x)\\ \text{from } a \text{ to } b\end{array} = A(b) - A(a)$$

$\int_a^b f(x)\,dx$ means

$$\lim_{\Delta x \to 0} \left(f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x \right).$$

$$= \lim_{\Delta x \to 0} \left(\text{sum of the areas of the little boxes } \underbrace{\square}_{\Delta x}\Big\} f(a+\Delta x)\right)$$



First box has area $f(a)\Delta x$

Second box has area $f(a+\Delta x)\Delta x$,

$\vdots$

Why is the fundamental theorem of calculus true?

$$\begin{array}{l}\text{Area under } f(x)\\ \text{from } a \text{ to } b\end{array} = A(b) - A(a)$$

where $\int f(x)\,dx = A(x) + c$.

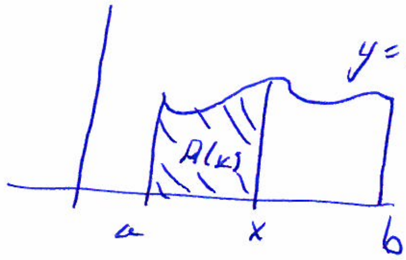
Idea of proof:

To show: There exists $A:[a,b]\to\mathbb{R}$ such that

(a) If $c\in[a,b]$ then $A'(c)=f(c)$

(b) $A(b)-A(a)=$ Area under $f(x)$ from a to b.

Let $A(x)=$ area under $f(x)$ from a to x



To show: (a) If $c\in[a,b]$ then $A'(c)=f(c)$

(b) $A(b)-A(a)=$ Area under $f(x)$ from a to b.

(a) Assume $c\in[a,b]$.

To show: $A'(c)=f(c)$.

$$A'(c)=\lim_{\Delta x\to 0}\frac{A(c+\Delta x)-A(c)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\begin{pmatrix}\text{area under } f(x)\\ \text{from } a \text{ to } c+\Delta x\end{pmatrix}-\begin{pmatrix}\text{area under } f(x)\\ \text{from } a \text{ to } c\end{pmatrix}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\begin{pmatrix}\text{area of little box}\\ \text{with height } f(c)\end{pmatrix}}{\Delta x}=\lim_{\Delta x\to 0}\frac{(f(c)\Delta x)}{\Delta x}$$

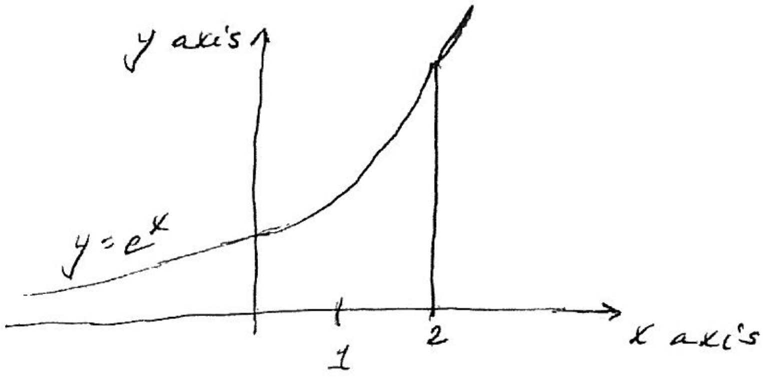
$$=\lim_{\Delta x\to 0}f(c)=f(c).$$

(b) To show: $A(b)-A(a)=$ area under $f(x)$ from a to b.

$$A(b) - A(a) = \begin{pmatrix} \text{area under } f(x) \\ \text{from } a \text{ to } b \end{pmatrix} - \begin{pmatrix} \text{area under } f(x) \\ \text{from } a \text{ to } a \end{pmatrix}$$

$$= \begin{pmatrix} \text{area under } f(x) \\ \text{from } a \text{ to } b \end{pmatrix}$$

<u>Example</u> $\int_0^2 e^x dx$



$$\int_0^2 e^x dx = \lim_{\Delta x \to 0} \left(e^0 \Delta x + e^{\Delta x} \Delta x + e^{2\Delta x} \Delta x + \cdots + e^{(2-\Delta x)} \Delta x \right)$$

If $\Delta x = \frac{1}{3}$, then

$$e^0 \Delta x + e^{\Delta x} \Delta x + \cdots + e^{2-\Delta x} \Delta x$$

$$= e^0 \tfrac{1}{3} + e^{\frac{1}{3}} \tfrac{1}{3} + e^{2/3} \tfrac{1}{3} + e^{3/3} \tfrac{1}{3} + e^{4/3} \tfrac{1}{3} + e^{5/3} \cdot \tfrac{1}{3}$$

$$= \tfrac{1}{3} \left(e^0 + e^{\frac{1}{3}} + (e^{\frac{1}{3}})^2 + (e^{\frac{1}{3}})^3 + (e^{\frac{1}{3}})^4 + (e^{\frac{1}{3}})^5 \right)$$

$$= \tfrac{1}{3} \left(\frac{e^{6/3} - 1}{e^{\frac{1}{3}} - 1} \right) = (e^2 - 1) \left(\frac{\frac{1}{3}}{e^{\frac{1}{3}} - 1} \right)$$

If $\Delta x = \frac{1}{5}$ then

$$e^0 \Delta x + e^{\Delta x} \Delta x + \cdots + e^{2-\Delta x} \Delta x = e^0 \tfrac{1}{5} + e^{\frac{1}{5}} \cdot \tfrac{1}{5} + \cdots + e^{9/5} \cdot \tfrac{1}{5}$$

$$= \tfrac{1}{5} \left(e^0 + e^{\frac{1}{5}} + (e^{\frac{1}{5}})^2 + (e^{\frac{1}{5}})^3 + \cdots + (e^{\frac{1}{5}})^9 \right)$$

$$= \tfrac{1}{5} \left(\frac{e^{10/5} - 1}{e^{\frac{1}{5}} - 1} \right) = (e^2 - 1) \frac{1/5}{e^{\frac{1}{5}} - 1}.$$

So

$$\int_0^2 e^x dx = \lim_{\Delta x \to 0} \left(e^0 \Delta x + \cdots + e^{2-\Delta x} \Delta x \right) = \lim_{N \to \infty} (e^2 - 1) \left(\frac{1/N}{e^{1/N} - 1} \right)$$

Recall: $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1.$

So $\lim\limits_{N \to \infty} \frac{e^{1/N} - 1}{1/N} = 1.$ So $\lim\limits_{N \to \infty} \frac{1/N}{e^{1/N} - 1} = 1.$

So

$$\int_0^2 e^x \, dx = \lim_{N \to \infty} (e^2 - 1)\left(\frac{1/N}{e^{1/N} - 1}\right) = (e^2 - 1) \cdot 1 = e^2 - 1.$$

Note: $\int e^x \, dx = e^x + c$ and

$$e^x + c \Big|_{x=0}^{x=2} = (e^2 + c) - (e^0 + c) = e^2 - 1.$$