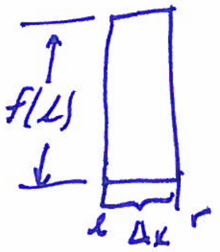
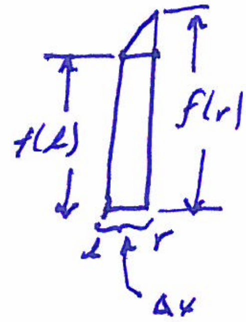
Areas

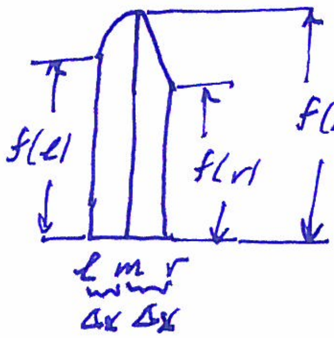


has area $f(l)\Delta x$



has area

$$f(l)\Delta x + \tfrac{1}{2}\Delta x\,(f(r) - f(l))$$
$$= \frac{\Delta x}{2}\,(2f(l) + f(r) - f(l))$$
$$= \frac{\Delta x}{2}\,(f(l) + f(r))$$

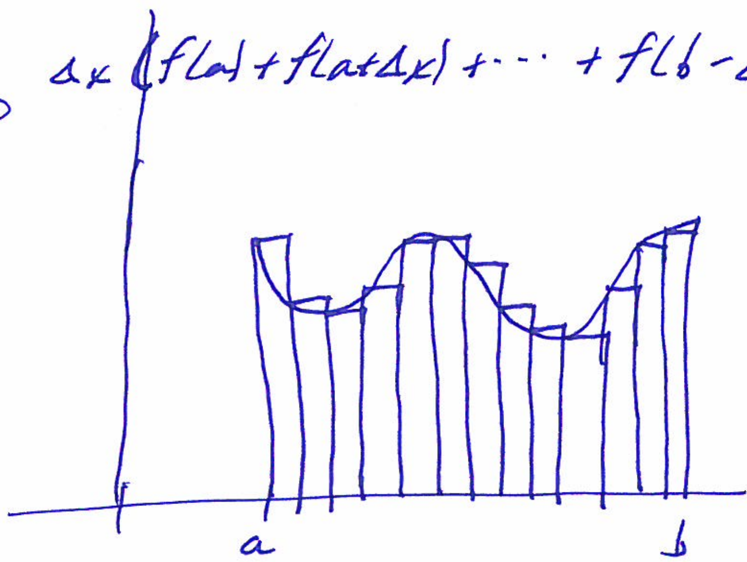


has area $\dfrac{\Delta x}{3}\,(f(l) + 4f(m) + f(r))$

Riemann Integral

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0}\left(\begin{array}{c}\text{sum of the areas of}\\ \text{the little rectangles}\end{array}\right)$$

$$= \lim_{\Delta x \to 0} \Delta x\,\big(f(a) + f(a+\Delta x) + \cdots + f(b - \Delta x)\big)$$



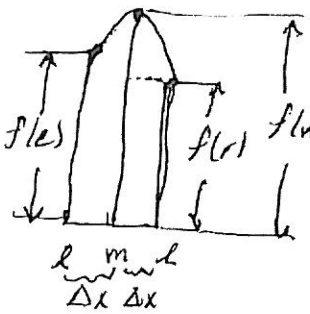
Trapezoidal integral

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \left(\begin{array}{l} \text{sum of the areas of} \\ \text{the little trapezoids} \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{2} \left(\begin{array}{l} f(a) + f(a+\Delta x) + f(a+\Delta x) + f(a+2\Delta x) + \\ + f(a+2\Delta x) + f(a+3\Delta x) + \cdots \\ \cdots + f(b-2\Delta x) + f(b-\Delta x) \\ + f(b-\Delta x) + f(b) \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{2} \left(\begin{array}{l} f(a) + 2f(a+\Delta x) + 2f(a+2\Delta x) + \cdots \\ \cdots + 2f(b-\Delta x) + f(b) \end{array} \right)$$

Simpson's Integral



has area $\frac{\Delta x}{3}\left(f(l) + 4f(m) + f(r)\right)$

so that

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \left(\begin{array}{l} \text{sum of the areas of} \\ \text{the little camel humps} \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{3} \left(\begin{array}{l} f(a) + 4f(a+\Delta x) + f(a+2\Delta x) \\ + f(a+2\Delta x) + 4f(a+3\Delta x) + f(a+4\Delta x) \\ + \cdots \\ + f(b-2\Delta x) + 4f(b-\Delta x) + f(b) \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{3} \left(f(a) + 4f(a+\Delta x) + 2f(a+2\Delta x) + 4f(a+3\Delta x) + 2f(a+4\Delta x) + \cdots + f(b) \right)$$

<u>Theorem</u> Let $N \in \mathbb{Z}_{>0}$ and $f:[a,b] \to \mathbb{R}$ be a function and $M \in \mathbb{R}_{>0}$.

(a) Assume that $f^{(N+1)}:[a,b] \to \mathbb{R}$ exists

and $|f^{(N+1)}(c)| < M$ for $c \in [a,b]$.

Let

$$\mathrm{TaylorErr}(N) = f(b) - \left(f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \cdots + \frac{1}{N!} f^{(N)}(a)(b-a) \right)$$

Then

$$|\mathrm{TaylorErr}(N)| < \frac{1}{(N+1)!} (b-a)^N M$$

(b) Assume that $f'':[a,b] \to \mathbb{R}$ exists and

$|f''(c)| < M$ for $c \in [a,b]$.

Let $\Delta x = \frac{b-a}{N}$ and

$$\mathrm{TrapErr}(N) = \int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)\frac{1}{2}\begin{pmatrix} f(a) + 2f(a+\Delta x) \\ + 2f(a+2\Delta x) + \cdots \\ \cdots + 2f(b-\Delta x) + f(b) \end{pmatrix}$$

Then

$$|\mathrm{TrapErr}(N)| < \frac{(b-a)^3}{12N^2} \cdot M$$

(c) Assume that $f^{(4)}:[a,b] \to \mathbb{R}$ exists and

$|f^{(4)}(c)| < M$ for $c \in [a,b]$.

Let $\Delta x = \frac{b-a}{N}$ and ④

$$\mathrm{SimpErr}(N) = \int_a^b f(x)\,dx - \frac{(b-a)}{3N}\Big(f(a) + 4f(a+\Delta x) + 2f(a+2\Delta x) + 4f(a+3\Delta x) + 2f(a+4\Delta x) + \cdots \cdots + 2f(b-2\Delta x) + 4f(b-\Delta x) + f(b) \Big)$$

Then

$$|\mathrm{SimpErr}(N)| < \frac{(b-a)^5}{180N^4} \cdot M$$

Example Find $\log(2)$ to within $.01$.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^N}{N} + \text{Error}(N).$$

In fact, if $f(x) = \log(1+x)$ then

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f^{(3)}(x) = \frac{(-1)(-2)}{(1+x)^3}$$

$$f^{(4)}(x) = \frac{(-1)(-2)(-3)}{(1+x)^4}, \ \ldots, \ f^{(N+1)}(x) = \frac{(-1)(-2)\cdots(-N)}{(1+x)^{N+1}}.$$

So $\text{TaylorError}(N) = \frac{(-1)(-2)\cdots(-N)}{(1+c)^{N+1}} \cdot \frac{1}{(N+1)!} \cdot (2-1)^{N+1}$ with $c \in (0,1)$.

So $|\text{TayErr}(N)| = \frac{1}{(N+1)(1+c)^{N+1}} < \frac{1}{N+1}.$

So, to get an approximation within $\frac{1}{100} = .01$
we should let $N = 99$.

So $\log(2) \simeq 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{97} - \frac{1}{98} + \frac{1}{99}$, to within $.01$.

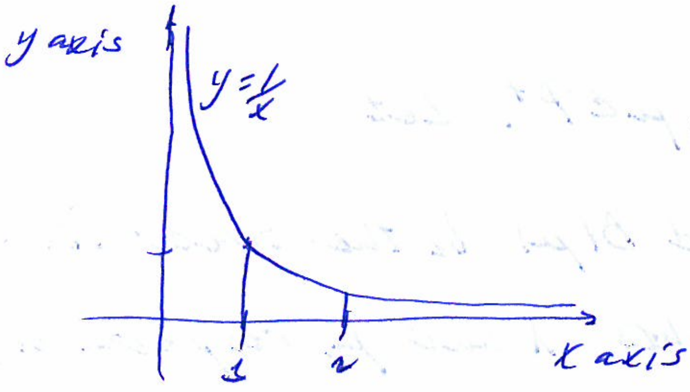
Example Find $\log 2$ to within $.01$.

$$\log 2 = \int_1^2 \frac{1}{x}\,dx$$

Use a trapezoidal approximation with

$f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$.

How many slices (i.e. what should N be)?



Then $f'(x)=\frac{-1}{x^2}$ and $f''(x)=\frac{(-1)(-2)}{x^3}=\frac{2}{x^3}$.

So $|f''(c)|<2.$ for $c\in[1,2]$

So $|\text{TrapError}(N)|<\frac{(b-a)^3}{12N^2}\cdot 2=\frac{1}{6\cdot N^2}$

If $N=5$ then $\frac{1}{6\cdot N^2}=\frac{1}{6\cdot 25}=\frac{1}{150}<.01.$

So 5 slices will do.

$$\log(2)=\int_1^2 \frac{1}{x}\,dx$$

$$\approx \frac{(b-a)}{5}\frac{1}{2}\Big(f(a)+2f(a+\Delta x)+2f(a+2\Delta x)+2f(a+3\Delta x)+2f(a+4\Delta x)+f(b)\Big)$$

$$=\frac{1}{10}\left(\frac{1}{1}+2\frac{1}{1+\frac{1}{5}}+2\frac{1}{1+\frac{2}{5}}+2\frac{1}{1+\frac{3}{5}}+2\frac{1}{1+\frac{4}{5}}+\frac{1}{2}\right)$$

$$=\frac{1}{10}\left(1+2\cdot\frac{5}{6}+2\cdot\frac{5}{7}+2\cdot\frac{5}{8}+2\cdot\frac{5}{9}+\frac{1}{2}\right)$$

$$=\frac{1}{10}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{20}.$$