Problem Set - Integral aproximations and the Fundamental Theorem of Calculus
620-295 Semester I 2010

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1. The Fundamental Theorem of Calculus

(1) What does

$$\int_a^b f(x)dx$$

mean?

(2) How does one usually calculate

$$\int_a^b f(x)dx \ ?$$

Give an example which shows that this method does not always work. Why doesn't it?

(3) Give an example which shows that

$$\int_a^b f(x)dx$$

is not always the true area under $f(x)$ between a and b even if $f(x)$ is continuous between a and b.

(4) What is the *Fundamental Theorem of Calculus*?

(5) Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

(6) Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

(7) Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this, compute an area by summing up the areas of tiny boxes and then show that applying

the Fundamental Theorem of Calculus gives the same result.

(8) Draw a picture illustrating the identity $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$. The fundamental theorem of calculus should indicate that there could be a corresponding identity for derivatives. What could it be?

(9) Draw a picture illustrating the identity $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ The fundamental theorem of calculus indicates that there should be a corresponding identity for derivatives. What is it?

(10) The fundamental theorem of calculus indicates that there should be an identity for integrals corresponding to the product rule for derivatives. What is it?

(11) Draw a picture illustrating the identity, if α is a constant then $\int_a^b \alpha f(x)dx = \alpha f(x)dx$. The fundamental theorem of calculus indicates that there should be a corresponding identity for derivatives. What is it?

(12) Draw a picture illustrating the following property in terms of areas: if $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$. The fundamental theorem of calculus indicates that there should be a corresponding identity for derivatives. What is it?

2. Riemann, Trapezoidal and Simpson approximations

(1) Define the Riemann integral.

(2) Define the trapezoidal integral.

(3) Define Simpson's integral.

(4) Determine the area of a trapezoid with left edge at $x = l$, right edge at $x = l + \Delta x$, left height $f(l)$, and right height $f(l + \Delta x)$.

(5) Determine the area of a parabola topped slice with left edge at $x = l$, right edge at $x = l + 2\Delta x$, middle at $x = l + \Delta x$, left height $f(l)$, middle height $f(l + \Delta x)$, and right height $f(l + 2\Delta x)$.

(6) Let N be a positive integer. Show that adding up N trapezoidal slices gives the approximation to $\int_a^b f(x)\, dx$ given by

$$\frac{\Delta x}{2}(f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \ldots + 2f(b - \Delta x) + f(b)),$$

where $\Delta x = \dfrac{b-a}{N}$.

(7) Let N be an even positive integer. Show that adding up N parabola topped slices gives the approximation to $\int_a^b f(x)\, dx$ given by

$$\frac{\Delta x}{2}(f(a) + 4f(a + \Delta x) + 2f(a + 2\Delta x) + \ldots + 4f(b - \Delta x) + f(b)),$$

where $\Delta x = \dfrac{b-a}{N}$.

(8) Let T_4 be the trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$. Show that $|f''(x)| \leq 2$ for $x \in [0,\ 1]$ and that $|T_4 - \frac{1}{4}\pi| \leq 1/96 < 0.0105$.

(9) Use the trapezoidal approximation with $N = 4$ slices to approximate the integral $\log 2 = \int_1^2 x^{-1}\, dx$. Show that $0.6866 \leq \log 2 \leq 0.6958$.

(10) Use Simpson's approximation with $N = 4$ slices to approximate $\log 2$. Show that $0.6927 \leq \log 2 \leq 0.6933$.

(11) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x^2}\, dx$.

(12) Compute a trapezoidal approximation with $N = 16$ slices for the integral $\int_0^1 e^{-x^2}\, dx$.

(13) Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x^2}\, dx$.

(14) Compute a Simpson approximation with $N = 16$ slices for the integral $\int_0^1 e^{-x^2}\, dx$.

(15) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$.

(16) Compute a trapezoidal approximation with $N = 16$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$.

(17) Compute a Simpson approximation with $N = 8$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$.

(18) Compute a Simpson approximation with $N = 16$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$.

(19) How many terms of the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ are needed to find the sum to an accuracy of 0.001?

(20) How many terms of the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n}}{n!}$ are needed to find the sum to an accuracy of 0.01?

(21) Estimate the integral $\int_0^1 \sin x^2\, dx$ to three decimal places by integrating the Maclaurin series for $\sin x^2$. Use the remainder estimate for alternating series to justify the result.

(22) The integral $\int_0^1 \frac{\sin x}{x}\, dx$ is difficult to approximate using, for example, left Riemann sums or the trapezoidal rule because the integrand $\frac{\sin x}{x}$ is not defined at $x = 0$. However, this integral converges, its value is approximately 0.94608.... Estimate the integral using Tyalor polynomials for $\sin x$ about $x = 0$ of degree 5.

(23) Derive the *midpoint approximation* for $\displaystyle\int_a^b f(x)\, dx$. With N slices it is obtained by adding up the areas of rectangles with height equal to the value of the function at the midpoint of

the interval. Show that the error estimate is given by $\left| \int_a^b f(x)\,dx - M_N \right| \le \frac{(b-a)^3}{24n^2} M$, where M is an upper bound for $|f''(x)|$ on $[a, b]$.

3. Additional problems -- NOT required for Assignment 4, 2010.

(1) Use the mean value theorem to prove that if f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), witha a bounded derivative, then f is Riemann integrable on $[a, b]$.

(2) Is $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases}$ integrable on $[0, 1]$?

(3) Give an example of a function that is bounded on the interval [0,1] but is not Riemann integrable on that interval.

(4) Explain carefully why you know that the function f defined by $f(t) = e^{\lfloor t \rfloor}$ is Riemann integrable on the interval [0,1000].

(5) Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^2 (1 + x^2)dx$ and obtain a bound for the error.

(6) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^2 (1 + x^2)dx$ and obtain a bound for the error.

(7) Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.

(8) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.

(9) Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^{\pi/2} \sin x\,dx$ and obtain a bound for the error.

(10) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^{\pi/2} \sin x\,dx$ and obtain a bound for the error.

(11) Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

(12) Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

(13) Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^2 (1 + x^2)dx$ and obtain a bound for the error.

(14) Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^2 (1+x^2)dx$ and obtain a bound for the error.

(15) Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.

(16) Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.

(17) Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^{\pi/2} \sin x\,dx$ and obtain a bound for the error.

(18) Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^{\pi/2} \sin x\,dx$ and obtain a bound for the error.

(19) Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^1 \left(1+x^2\right)^{-1} dx$ and obtain a bound for the error.

(20) Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 \left(1+x^2\right)^{-1} dx$ and obtain a bound for the error.

4. References

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