Real Analysis with Applications Lecture 18, 19.04.2010

Derivatives by limits

Let $f : [a,b] \to \mathbb{R}$. Let $c \in [a,b]$.

The derivative of f at $x = c$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(a)}{x - c}$$

Alternatively,

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(x)}{\Delta x}$$

Theorems Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ and let $\beta, \gamma \in \mathbb{R}$. Assume that $f'(c)$ and $g'(c)$ exist. Then

(a) $(\beta f + \gamma g)'(c) = \beta f'(c) + \gamma g'(c)$,

(b) $(fg)'(c) = f'(c) g(c) + f(c) g'(c)$,

(c) Assume $f : [a,b] \to \mathbb{R}$ is given by $f(x) = x$. then $f'(c) = 1$.

(d) If $f'(c)$ exists then f is continuous at $x = c$.

Define

$$f''(c) = (f')'(c), \quad f^{(3)}(c) = (f'')'(c) \quad \text{and}$$

$$f^{(N)}(c) = (f^{(N-1)})'(c).$$

<u>Theorem</u> (Taylor's theorem with Lagrange's remainder).

If $f:[a,b]\to\mathbb{R}$ and $N\in\mathbb{Z}_{\geq 0}$ and $f^{(N)}:[a,b]\to\mathbb{R}$ is continuous and $f^{(N+1)}:[a,b]\to\mathbb{R}$ exists then there exists $c\in(a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \cdots$$

$$\cdots + \frac{1}{N!}f^{(N)}(a)(b-a)^N + \frac{1}{(N+1)!}f^{(N+1)}(c)(b-a)^{N+1}$$

<u>Remarks:</u>

(1) The last term in $f(b)=f(a)+\cdots+\frac{1}{(N+1)!}f^{(N+1)}(c)(b-a)^{N+1}$ is <u>Lagrange's form of the remainder</u>.

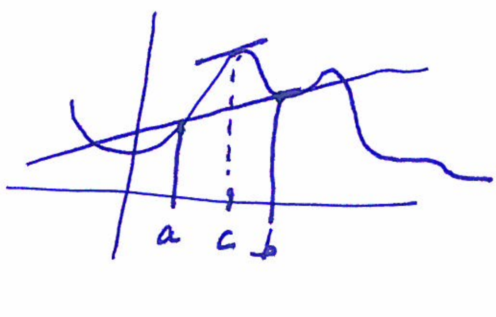
(2) The special case $N=0$ is the <u>Mean Value Theorem</u>

If $f:[a,b]\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ is continuous and $f':[a,b]\to\mathbb{R}$ exists then there exists $c\in(a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$

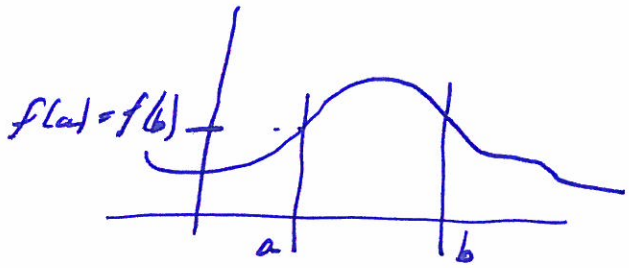
i.e.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



(3) The special case $N=0$ and $f(a)=f(b)$ is <u>Rolle's</u> <u>theorem</u>:

If $f:[a,b]\to\mathbb{R}$ is continuous and $f':[a,b]\to\mathbb{R}$ exists then there exists $c\in[a,b)$ such that $f'(c)=0$



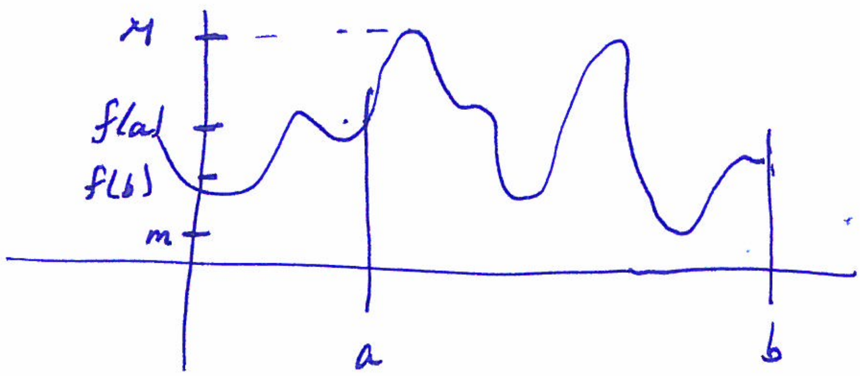
(4) The proof of these theorems uses

<u>Theorem</u> (Intermediate value theorem)

(a) If $f:[a,b]\to\mathbb{R}$ is continuous and w is between $f(a)$ and $f(b)$ then there exists $c\in(a,b)$ such that $f(c)=w$.

(b) If $f:[a,b]\to\mathbb{R}$ is continuous then there exist $m, M\in\mathbb{R}$ such that

$$f([a,b])=[m,M]$$



(5) Let $f:[0,2\pi] \to \mathbb{C}$ be given by

$$f(x) = \cos x + i \sin x$$

Then $f(0) = f(2\pi)$ but $f'(x)$ is never 0.
Why is this not a contradiction to Rolle's theorem?

(6) The first N terms in (all but the remainder term)

$$f(b) = f(a) + \cdots + \frac{1}{N!} f^{(N)}(a)(b-a)^N + \frac{1}{(N+1)!} f'(c)(b-a)^{N+1}$$

are the <u>Taylor approximation to f at $x=a$ of order N</u>.

(7) $f:[a,b] \to \mathbb{R}$ is <u>differentiable at $x=c$</u> if $f'(c)$ exists.

<u>Example</u> Approximate $28^{\frac{1}{3}}$ to 5 decimal places

Let $f(x) = (27+x)^{\frac{1}{3}}$. Then

$$(27+x)^{\frac{1}{3}} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

So $a_0 = (27+0)^{\frac{1}{3}} = 3.$

$$a_1 = \left(\frac{d}{dx}(27+x)^{\frac{1}{3}}\right)\Big|_{x=0} = \frac{1}{3}(27+x)^{-2/3}\Big|_{x=0} = \frac{1}{3}\frac{1}{3^2} = \frac{1}{27}$$

$$a_2 = \frac{1}{2!}\left(\frac{d^2}{dx^2}(27+x)^{\frac{1}{3}}\right)\Big|_{x=0} = \frac{1}{2}\frac{1}{3}\left(\frac{-2}{3}\right)(27+x)^{-5/3}\Big|_{x=0} = \frac{1}{2}\cdot\frac{(-2)}{3^2}\frac{1}{3^5}$$

$$= -\frac{1}{3^7}$$

$$a_3 = \frac{1}{3!}\left(\frac{d^3}{dx^3}(27+x)^{\frac{1}{3}}\right)\Bigg|_{x=0} = \frac{1}{2\cdot 3}\ \frac{1}{3}\,\frac{(-2)}{3}\,\frac{(-5)}{3}\,(27+x)^{-8/3}\Bigg|_{x=0}$$ (5)

$$= \frac{5}{3^4}\cdot\frac{1}{3^8} = \frac{5}{3^{12}}$$

$$\frac{1}{4!} f^{(4)}(c)(28-27)^4 = \frac{1}{4!}\ \frac{1}{3}\,\frac{(-2)}{3}\,\frac{(-5)}{3}\,\frac{(-8)}{3}\,(27+c)^{-11/3}$$

$$= \frac{-2^4\cdot 5}{2^3\cdot 3^5}(27+c)^{-11/3} = \frac{-2\cdot 5}{3^5}\,\frac{1}{(27+c)^{11/3}}$$

So $(27+x)^{\frac{1}{3}} = 3 + \frac{1}{27}x - \frac{1}{3^7}x^2 + \frac{5}{3^{12}}x^3 + \dots$

and $28^{\frac{1}{3}} = (27+1)^{\frac{1}{3}} =$

$$\simeq 3 + \frac{1}{27} - \frac{1}{3^7} + \frac{5}{3^{12}}$$

with error equal to

$$\frac{2\cdot 5}{3^5}\,\frac{1}{(27+c)^{11/3}}$$ for some ~~$c \in (27, 28)$~~ $c \in (0,1)$

So the error is less than

$$\frac{2\cdot 5}{3^5}\,\frac{1}{27^{11/3}} = \frac{2\cdot 5}{3^5\cdot 3^{11}} = \frac{2\cdot 5}{3^{16}}.$$