620-295 Real Analysis Lecture 17, 16 April 2010

Theorem 1 Let (a_n) be a sequence in $\mathbb{R}$.

(a) Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a<1$

Then $\sum_{n=1}^{\infty} |a_n|$ converges.

(b) Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a>1$

Then $\sum_{n=1}^{\infty} |a_n|$ diverges

Theorem 2 Let (a_n) be a sequence in $\mathbb{R}$.

(a) Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a<1$

Then $\sum_{n=1}^{\infty} |a_n|$ converges

(b) Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a>1$

Then $\sum_{n=1}^{\infty} |a_n|$ diverges.

Proof of theorem 1a

Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ and $a<1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $a < a+\varepsilon < 1$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$\left| \frac{|a_{n+1}|}{|a_n|} - a \right| < \varepsilon.$$

Then

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$= |a_1| + |a_2| + \cdots + |a_N| + |a_N| \cdot \frac{|a_{N+1}|}{|a_N|} + |a_N| \frac{|a_{N+1}|}{|a_N|} \frac{|a_{N+2}|}{|a_{N+1}|} + \cdots$$

$$\leq |a_1| + |a_2| + \cdots + |a_N| + |a_N| \left(\frac{|a_{N+1}|}{|a_N|} + \frac{|a_{N+1}|}{|a_N|} \frac{|a_{N+2}|}{|a_{N+1}|} + \frac{|a_{N+1}|}{|a_N|} \frac{|a_{N+2}|}{|a_{N+1}|} \cdot \frac{|a_{N+3}|}{|a_{N+2}|} + \cdots \right)$$

$$\leq |a_1| + \cdots + |a_N| + |a_N| \left((a+\varepsilon) + (a+\varepsilon)^2 + (a+\varepsilon)^3 + \cdots \right)$$

$$= |a_1| + \cdots + |a_N| + |a_N| \left(\frac{1}{1-(a+\varepsilon)} \right)$$

So the ratio test is a comparison to a geometric series!

Proof of theorem 2a

Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ and $a<1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $a < a+\varepsilon < 1$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then

$$||a_n|^{1/n} - a| < \varepsilon.$$

Then

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$= |a_1| + \cdots + |a_N| + \left(|a_{N+1}|^{\frac{1}{N+1}}\right)^{N+1} + \left(|a_{N+2}|^{\frac{1}{N+2}}\right)^{N+2} + \cdots$$

$$\le |a_1| + \cdots + |a_N| + (a+\varepsilon)^{N+1} + (a+\varepsilon)^{N+2} + \cdots$$

$$= |a_1| + \cdots + |a_N| + (a+\varepsilon)^{N+1}\left(1 + (a+\varepsilon) + (a+\varepsilon)^2 + \cdots\right)$$

$$= |a_1| + \cdots + |a_N| + (a+\varepsilon)^{N+1}\left(\frac{1}{1-(a+\varepsilon)}\right).$$

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A sequence (a_n) is contractive if there exists $\alpha \in \mathbb{R}$, $\alpha \in (0,1)$ such that

$$|a_{n+1} - a_n| \le \alpha\, |a_n - a_{n-1}| \quad \text{for } n = 2, 3, 4, \ldots$$

If (a_n) is contractive then

$$
\begin{aligned}
|a_{n+1}-a_n| &\le \alpha\,|a_n-a_{n-1}| \\
&\le \alpha^2\,|a_{n-1}-a_{n-2}| \\
&\le \alpha^3\,|a_{n-2}-a_{n-3}| \\
&\le \\
&\ \ \vdots \\
&\le \alpha^{n-1}\,|a_{n-(n-2)}-a_{n-(n-1)}| \\
&\le \alpha^{n-1}\,|a_2-a_1|,
\end{aligned}
$$

which is very small if $n=1000000$ and $\alpha=\frac{1}{2}$.

This is the idea behind the proof of the ratio test.

A sequence (a_n) is <u>Cauchy</u> if (a_n) satisfies:

If $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that

if $m,n\in\mathbb{Z}_{>0}$ ~~then~~ and $m,n>N$ then

$$d(a_m,a_n)<\varepsilon.$$

<u>Proposition</u> There does not exist $\frac{a}{b}\in\mathbb{Q}$ such that $\left(\frac{a}{b}\right)^2=2$.

<u>Proof</u> Proof by contradiction.

Assume $\frac{a}{b}\in\mathbb{Q}$ and $\frac{a^2}{b^2}=2$. and $\frac{a}{b}$ is reduced.

Then $a^2=2b^2$, so that a^2 is even.

So a is even.

$\therefore$ $2b^2$ is divisible by 4.

$\therefore$ b^2 is even.

$\therefore$ b is even.

$\therefore$ $\frac{a}{b}$ is not reduced.

Contradiction.

$\therefore$ there does not exist $\frac{a}{b} \in \mathbb{Q}$ with $\left(\frac{a}{b}\right)^2 = 2$. //

Consider the sequence in $\mathbb{Q}$:

$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots\right)$$

This is a Cauchy sequence that does not converge.

Consider the sequence in $\mathbb{R}$:

$$(1.00\ldots, 1.4000,\ldots, 1.4100\ldots, 1.41400\ldots, 1.414200,\ldots)$$

This is a Cauchy sequence that does converge.