Additional Radius of convergence examples 15.04.2010

Example Find the radius of convergence of the Taylor series for e^x at the point $a=-3$.

We want e^x expanded in powers of $x+3$.

$$e^x = e^{-3}e^{x+3} = e^{-3}\left(1+(x+3)+\frac{(x+3)^2}{2!}+\frac{(x+3)^3}{3!}+\cdots\right)$$

For which values of x does this series converge?

Let $y=x+3$. Then

$$e^x = e^{-3}\left(1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots\right)$$

For which values of y does this series converge?

Apply the ratio test:

$$\lim_{n\to\infty}\frac{\frac{y^{n+1}}{(n+1)!}}{\frac{y^n}{n!}} = \lim_{n\to\infty}\frac{y}{n+1} = 0, \text{ which is always } < 1.$$

So the series converges for all values of y.

So the series converges for all values of x (since $y=x+3$).

<u>Example</u> Find the radius of convergence of the power series expansion of $\int \frac{\sinh x}{x}\,dx$.

$$\int \frac{\sinh x}{x}\,dx = \int \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x}\,dx$$

$$= \int \left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right) dx$$

$$= x + \frac{1}{3}\frac{x^3}{3!} + \frac{1}{5}\frac{x^5}{5!} + \frac{1}{7}\frac{x^7}{7!} + \cdots$$

Apply the ratio test:

$$\lim_{n\to\infty} \frac{\dfrac{x^{2(n+1)+1}}{(2n+3)(2n+3)!}}{\dfrac{x^{2n+1}}{(2n+1)(2n+1)!}} = \lim_{n\to\infty} \frac{(2n+1)\,x^2}{(2n+3)(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{(2+\frac{1}{n})\,x^2}{(2+\frac{3}{n})(2n+3)(2n+2)} = 0,$$ which is always < 1. So the series converges for <u>all</u> x.

<u>Example</u> Find the radius of convergence of the power series expansion of $\sinh x$.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

For which values of x does this series converge?

Apply the ratio test:

$$\lim_{n\to\infty} \frac{\dfrac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\dfrac{x^{2n+1}}{(2n+1)!}} = \lim_{n\to\infty} \frac{\dfrac{x^{2n+3}}{(2n+3)!}}{\dfrac{x^{2n+1}}{(2n+1)!}} = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0$$

which is less than 1 for all values of x.

So the series converges for <u>all</u> values of x.

<u>Example</u> Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

This series doesn't depend on x so the question doesn't really make sense. Probably the question should be,

"Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{n}{2^n} x^n.$$

For which x does this series converge?

Apply the ratio test:

$$\lim_{n\to\infty} \frac{\left|\dfrac{(n+1)x^{n+1}}{2^{n+1}}\right|}{\left|\dfrac{nx^n}{2^n}\right|} = \lim_{n\to\infty} \left|\frac{(n+1)}{n}\frac{x}{2}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\frac{|x|}{2}$$

$$= \frac{|x|}{2}.$$

This is <1 if $|x|<2$. So the series converges if $-2 < x < 2$.

So the radius of convergence is 2 and if $x \in \mathbb{R}$ the interval of convergence is $(-2, 2)$ or $[-2, 2)$ or $(-2, 2]$ or $[-2, 2]$.

If $x = 2$ the series is

$$\sum_{n=1}^{\infty} \frac{n\,2^n}{2^n} = \sum_{n=1}^{\infty} n, \text{ which diverges}$$

If $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{n\,(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n = -1 + 2 - 3 + 4 - \cdots,$$

which diverges.

So the interval of convergence is $(-2, 2)$.

<u>Example</u> Find the radius and interval of convergence of

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k x^k}{k}$$

For which x does this series converge?

Apply the ratio test:

$$\lim_{k\to\infty} \frac{\left| \dfrac{(-1)^{k+1+1}\, 2^{k+1} x^{k+1}}{k+1} \right|}{\left| \dfrac{(-1)^{k+1}\, 2^k x^k}{k} \right|} = \lim_{k\to\infty} \left| \frac{2x\,k}{k+1} \right|$$

$$= \lim_{k\to\infty} 2|x| \left(\frac{1}{1+\frac{1}{k}} \right) = 2|x|$$

which is <1 if $|x| < \frac{1}{2}$.

So the radius of convergence is $\frac{1}{2}$ and if $x \in \mathbb{R}$ the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$ or $\left[-\frac{1}{2}, \frac{1}{2}\right)$ or $\left(-\frac{1}{2}, \frac{1}{2}\right]$ or $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

If $x = \frac{1}{2}$ the series is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k \frac{1}{2^k}}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which converges to $\log(1+1) = \log 2$.

If $x = -\frac{1}{2}$ the series is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k \left(\frac{-1}{2}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)}{k} = -\sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges as $\sum_{k=1}^{\infty} \frac{1}{k}$ is a harmonic series.

So the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

<u>Example</u> Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n}}$.

Let $y = 2x-1$. Then the series is $\displaystyle\sum_{n=1}^{\infty} \frac{y^n}{n^{3/2}}$.

For which y does this series converge?

Apply the ratio test:

$$\lim_{n\to\infty} \frac{\left|\frac{y^{n+1}}{(n+1)^{3/2}}\right|}{\left|\frac{y^n}{n^{3/2}}\right|} = \lim_{n\to\infty} |y|\left(\frac{n}{n+1}\right)^{3/2}$$

$$= \lim_{n\to\infty} |y| \frac{1}{\left(1+\frac{1}{n}\right)^{3/2}} = |y|,$$

which is <1 if $|y|<1$. So the series converges if $|y|<1$. So the series converges if $|2x-1|<1$. So if $x\in\mathbb{R}$ the series converges if

$$-1 < 2x-1 \text{ and } 2x-1 < 1.$$

So, if $x\in\mathbb{R}$ the series converges if

$$0 < 2x \text{ and } 2x < 2.$$

So, if $x\in\mathbb{R}$ the series converges if $0<x<1$.

So the radius of convergence is $\frac{1}{2}$ and the interval of convergence is

$(0,1)$ or $[0,1]$ or $(0,1]$ or $[0,1)$.

If $x=0$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

which converges by the Leibniz test since $\lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$

If $x=1$ the series is $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which

converges since it is a p-harmonic series
with $p = 3/2$ and $3/2 > 1$.

So the interval of convergence is $[0, 1]$.