Examples of improper integrals and series Lecture 16 14.04.2010

Example Analyse $\int_1^\infty \frac{\log x}{x}\,dx$

Let $y = \frac{\log x}{x} = x^{-1}\log x$. Then

$$\frac{dy}{dx} = x^{-1}\frac{1}{x} + (-1)x^{-2}\log x = (1-\log x)\frac{1}{x^2}$$

This is > 0 for $x < e^1$

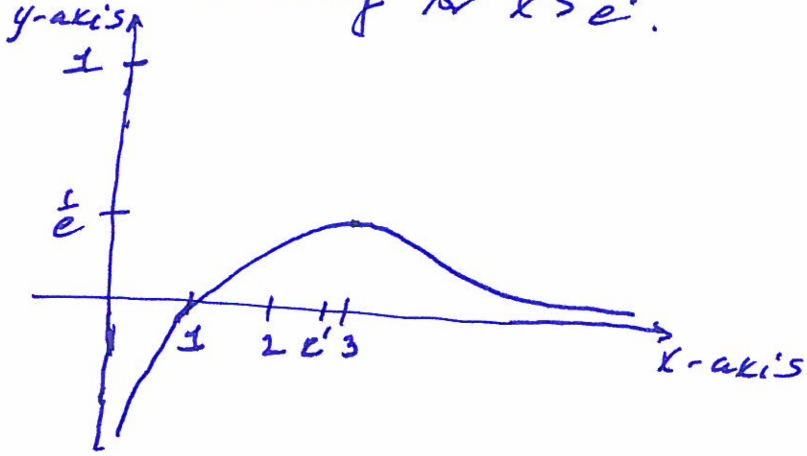
$= 0$ for $x = e^1$

< 0 for $x > e^1$.

So y is increasing for $x < e^1$,

has a maximum at $x = e^1 \simeq 2.781828\ldots$

is decreasing for $x > e^1$.



$$\int_1^\infty \frac{\log x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\log x}{x}\,dx$$

$$= \lim_{b\to\infty}\left(\frac{(\log x)^2}{2}\Big|_{x=1}^{x=b}\right) = \lim_{b\to\infty}\left(\frac{(\log b)^2}{2} - \frac{(\log 1)^2}{2}\right)$$

$=$ divergent.

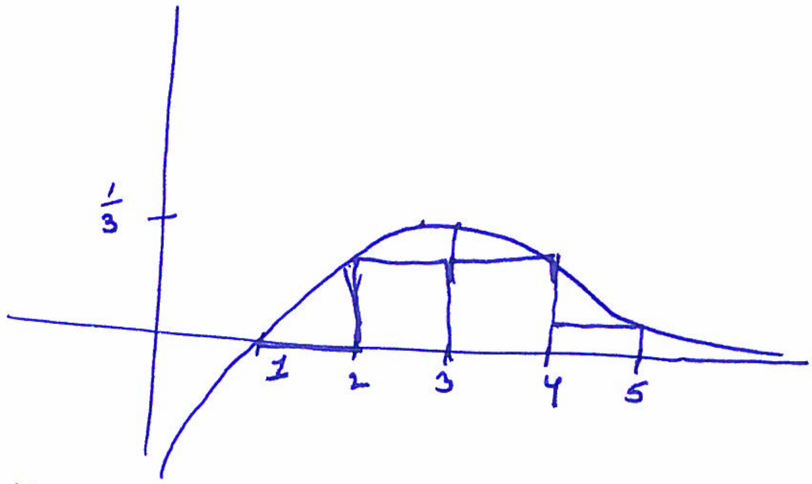
<u>Example</u> $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n} = -0 + \frac{\log 2}{2} - \frac{\log 3}{3} + \sum_{n=4}^{\infty} (-1)^n \frac{\log n}{n}$$

Since the values $\frac{\log n}{n}$ are positive, decreasing (see the previous page) and approaching 0 for $n \geq 4$, the alternating series test guarantees that

$\sum_{n=4}^{\infty} (-1)^n \frac{\log n}{n}$ converges.

So $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n} = \frac{\log 2}{2} - \frac{\log 3}{3} + \sum_{n=4}^{\infty} (-1)^n \frac{\log n}{n}$ converges.



$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\log n}{n} \right| = \sum_{n=1}^{\infty} \frac{\log n}{n} = \frac{\log 2}{2} + \frac{\log 3}{3} + \sum_{n=4}^{\infty} \frac{\log n}{n}$$

$$< \frac{\log 2}{2} + \frac{\log 3}{3} + \int_4^{\infty} \frac{\log n}{n}$$ diverges (from the previous page).

So $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$ is conditionally convergent but not absolutely convergent.

To apply the ratio test (to determine absolute convergence) one must analyse

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+1}\frac{\log n+1}{n+1}\right|}{\left|(-1)^{n}\frac{\log n}{n}\right|} = \lim_{n\to\infty} \frac{n\log(n+1)}{(n+1)\log n}$$

$$= \lim_{n\to\infty} \frac{n\left(\log\left(1+\frac{1}{n}\right)-\log n\right)}{(n+1)\left(\log\left(1+\frac{1}{n-1}\right)-\log(n-1)\right)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{n}{n}\right)\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} - n\log n}{\left(\frac{n+1}{n-1}\right)\frac{\log\left(1+\frac{1}{n-1}\right)}{1/n-1} - (n+1)\log(n-1)}$$

which is not very efficient. This makes sense since the ~~function~~ $\cancel{y = \frac{\log x}{x}}$ series $\sum_{n=1}^{\infty} \frac{\log n}{n}$ is not readily comparable to a series of the form $\sum_{n=1}^{\infty} a^n$ and ratio test is really a comparison to a series of this form.

Similarly the root test limit

$$\lim_{n\to\infty} \left(\frac{\log n}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{(\log n)^{\frac{1}{n}}}{n^{\frac{1}{n}}} = 1$$

is not helpful for determining convergence.

<u>Example</u> Evaluate $\lim\limits_{n\to\infty} n^{1/n}$.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \left(e^{\log n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log n}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\log(n-1+1)} = \lim_{n\to\infty} e^{\frac{1}{n}\left(\log\left(1+\frac{1}{n-1}\right)+\log(n-1)\right)}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n(n-1)}\left(\frac{\log\left(1+\frac{1}{n-1}\right)}{1/n-1}\right) + \frac{1}{n}\log(n-1)}$$

$$= e^{0\cdot 1+0} = e^{0} = 1.$$