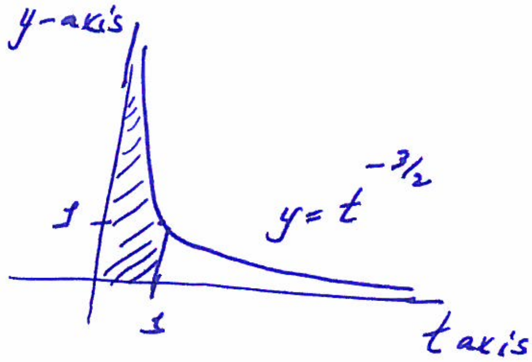
<u>Example</u>

$$\int_0^1 t^{-3/2}\,dt$$



$$= \lim_{b\to 0} \int_b^1 t^{-3/2}\,dt$$

$$= \lim_{b\to 0} \left(-2t^{-\frac{1}{2}} \Big|_{t=b}^{t=1} \right) = \lim_{b\to 0} \left(-2 - (-2b^{-\frac{1}{2}}) \right)$$

$$= \lim_{b\to 0} \left(-2 + \frac{2}{b^{\frac{1}{2}}} \right) = \text{diverges}.$$

<u>Example</u> Let $n \in \mathbb{Z}_{\geq 0}$. Show that $n! = \int_0^\infty e^{-t} t^n\,dt$

<u>Proof</u>. Proof by induction.

If $n = 0$ then

$$\int_0^\infty e^{-t} t^0\,dt = \int_0^\infty e^{-t}\,dt = \lim_{b\to\infty} \int_0^b e^{-t}\,dt = \lim_{b\to\infty} \left(-e^{-t} \Big|_{t=0}^{t=b} \right)$$

$$= \lim_{b\to\infty} \left(-e^{-b} - (-e^0) \right) = 0 + 1 = 1.$$

If $n=1$ then, since $\frac{d\,te^{-t}}{dt} = e^{-t} + -te^{-t}$

$$\int_0^b (-e^{-t} t + e^{-t})\,dt = \int_0^b \frac{d(te^{-t})}{dt}\,dt = te^{-t} \Big|_{t=0}^{t=b}$$

$$= be^{-b} - 0e^0 = be^{-b}$$

So

$$\int_0^b e^{-t} t\,dt = -be^{-b} + \int_0^b e^{-t}\,dt = -be^{-b} + \left(-e^{-t} \Big|_{t=0}^{t=b} \right)$$

$$= -be^{-b} + \left(-e^{-b} - (-e^0) \right) = -be^{-b} - e^{-b} + 1$$

So $\int_0^\infty e^{-t} t\,dt = \lim_{b\to\infty} \int_0^b e^{-t} t\,dt = \lim_{b\to\infty} \left(\frac{-b}{e^b} \right) - \frac{1}{e^b} + 1 = 0 - 0 + 1 = 1.$

Let $N \in \mathbb{Z}_{\geq 0}$. Assume that if $n \in \mathbb{Z}_{\geq 0}$ and $n < N$ then $n! = \int_0^\infty e^{-t} t^n \, dt$.

To show: $N! = \int_0^\infty e^{-t} t^N \, dt$.

Right hand side: $\int_0^\infty e^{-t} t^N \, dt = \lim_{b \to \infty} \int_0^b e^{-t} t^N \, dt$.

Since $\frac{d\, t^N e^{-t}}{dt} = N t^{N-1} e^{-t} - t^N e^{-t}$,

$$\int_0^b \left(N t^{N-1} e^{-t} - t^N e^{-t}\right) dt = \int_0^b \frac{d\, t^N e^{-t}}{dt} \, dt$$

$$= t^N e^{-t} \Big|_{t=0}^{t=b} = b^N e^{-b} - 0^N e^{-0} = \frac{b^N}{e^b}$$

So

$$\lim_{b \to \infty} \int_0^b \left(N t^{N-1} e^{-t} - t^N e^{-t}\right) dt = \lim_{b \to \infty} \frac{b^N}{e^b} = 0.$$

So

$$\lim_{b \to \infty} \int_0^b t^N e^{-t} \, dt = \lim_{b \to \infty} \int_0^b N t^{N-1} e^{-t} \, dt$$

$$= N \int_0^\infty t^{N-1} e^{-t} \, dt = N \cdot (N-1)!,$$

where the last equality is by the induction hypothesis.

So

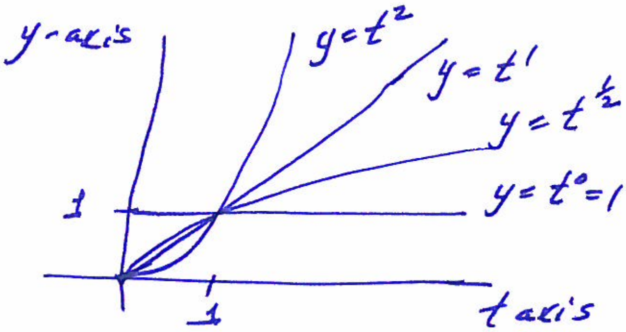
$$\int_0^\infty t^N e^{-t} \, dt = N (N-1)! = N!.$$

//

<u>Example</u> Let $m \in \mathbb{R}_{\le 0}$. Analyse $\int_0^\infty t^{-m}\,dt$.

Another way to state the question is:

Let $p \in \mathbb{R}_{\ge 0}$. Analyse $\int_0^\infty t^p\,dt$



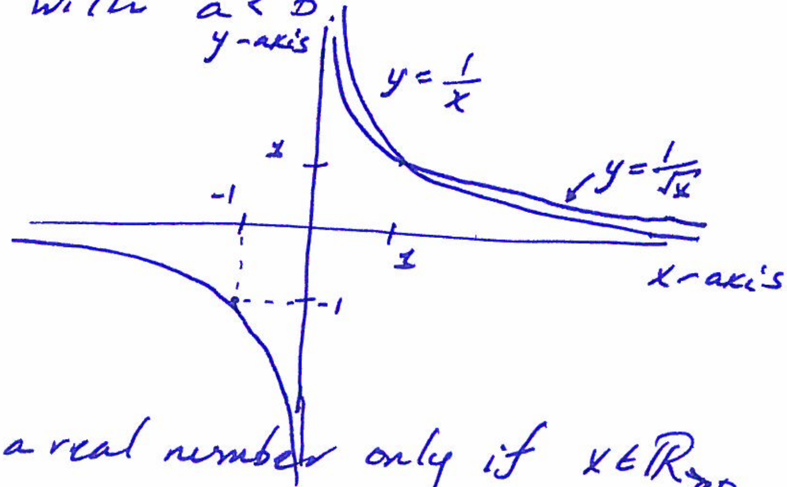
From the picture $\int_0^\infty t^p\,dt = \begin{pmatrix} \text{area under } y=t^p \\ \text{from } t=0 \text{ to } t=\infty \end{pmatrix}$

is an infinite area.

So $\int_0^\infty t^p\,dt$ diverges.

<u>Example</u> Let $a, b \in \mathbb{R}$ with $a<b$.

Analyze $\int_a^b x^{-\frac{1}{2}}\,dx$.

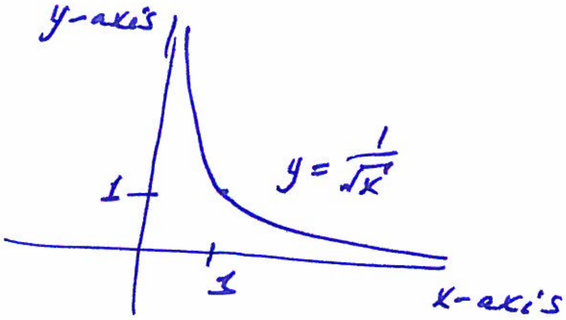


Note that $x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$ is a real number only if $x \in \mathbb{R}_{>0}$

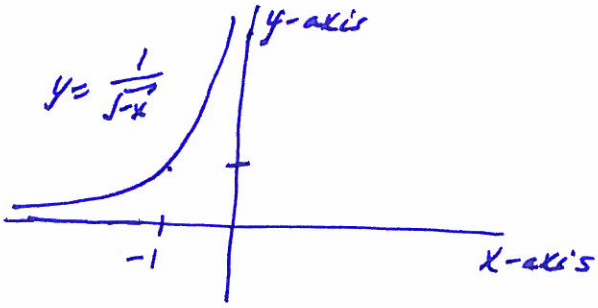
So a and b must be in $\mathbb{R}_{\ge 0}$ for the problem to make good sense.

$$\int_a^b x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_{x=a}^{x=b} = 2b^{\frac{1}{2}} - 2a^{\frac{1}{2}} = 2(\sqrt{b}-\sqrt{a}).$$

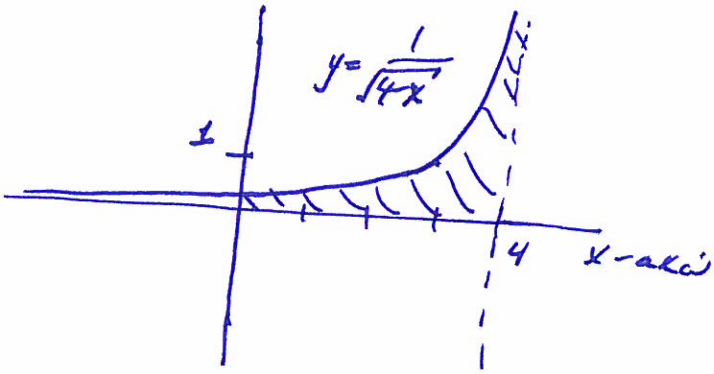
Example $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx.$



~~Let $z = 4-x = (-x)+4$.~~



and



$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4}\int_0^b \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4}\int_0^b (4-x)^{\frac{1}{2}}\,dx$$

$$= \lim_{b\to 4}\left(\frac{-2}{3}(4-x)^{3/2}\Big|_{x=0}^{x=b}\right) = \lim_{b\to 4}\left(\frac{-2}{3}(4-b)^{3/2} - \left(\frac{-2}{3}4^{3/2}\right)\right)$$

$$= 0 + \frac{2}{3}4^{3/2} = \frac{2}{3}\cdot 2^3 = \frac{16}{3}$$

Example $\displaystyle\int_1^{\infty} \frac{(\log x)^4}{1+x^2}\,dx$ Let $x = e^t$.

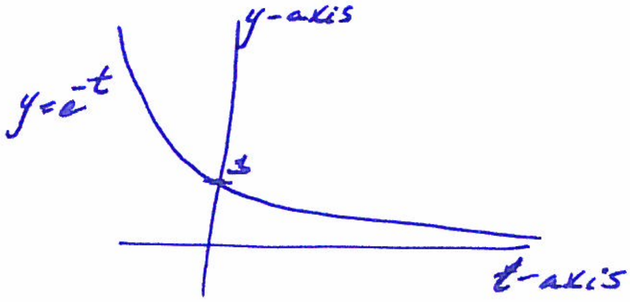
Then $\frac{dx}{dt} = e^t$. So $\displaystyle\int_{x=1}^{x=\infty} \frac{(\log x)^4}{1+x^2}\,dx = \int_{t=0}^{t=\infty} \frac{(\log e^t)^4}{1+(e^t)^2}\,\frac{dx}{dt}\,dt$

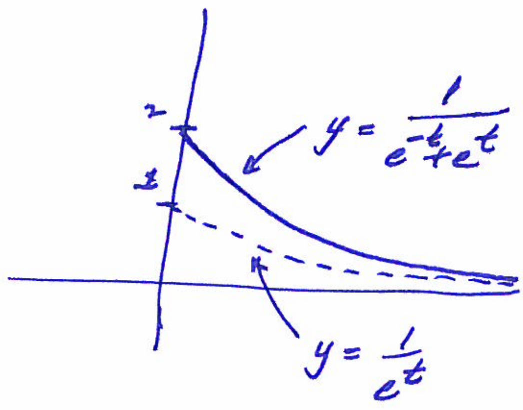
$$= \int_0^{\infty} \left(\frac{t^4}{1+e^{2t}}\right) e^t\,dt = \int_0^{\infty} \frac{t^4 e^t}{1+(e^t)^2}\,dt$$

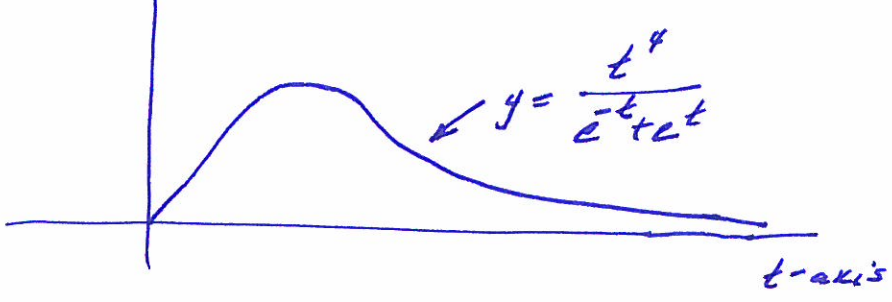
$$= \int_0^{\infty} \frac{t^4}{e^{-t}+e^t}\,dt < \int_0^{\infty} \frac{t^4}{e^t}\,dt = \int_0^{\infty} t^4 e^{-t}\,dt$$

This last integral is equal to 4! by problem 4 of section 4: Improper integrals: Analysis and Applications (the Gamma function in its integral form)

So $\displaystyle\int_1^{\infty} \frac{(\log x)^4}{1+x^2}\,dx$ converges.



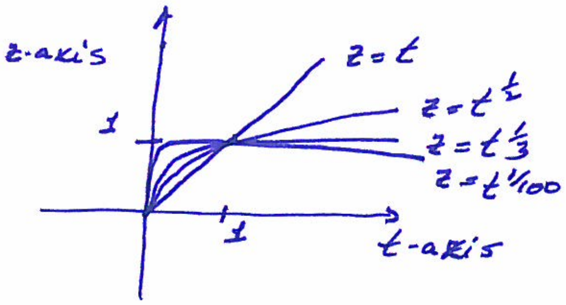


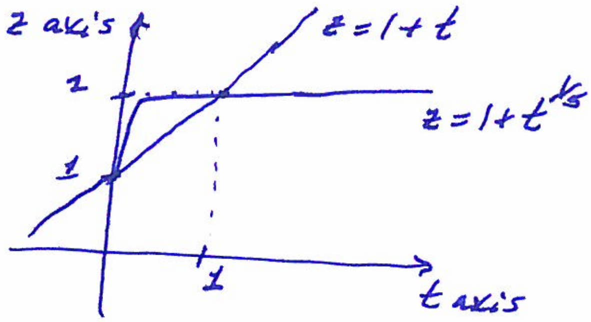


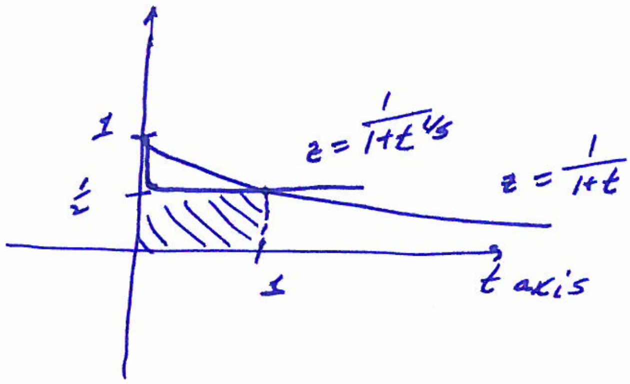
<u>Example</u> $\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim_{n\to\infty}\int_{y=0}^{y=1}\frac{1}{1+y}\,\frac{dy^n}{dy}\,dy$

Let $t = y^n$. Then

$$\int_{y=0}^{y=1}\frac{1}{1+y}\,\frac{dy^n}{dy}\,dy = \int_{t=0}^{t=1}\frac{1}{1+t^{\frac{1}{n}}}\,\frac{dt}{dy}\,dy = \int_{t=0}^{t=1}\frac{1}{1+t^{1/n}}\,dt$$







So $\lim_{n\to\infty}\int_{t=0}^{t=1}\frac{1}{1+t^{1/n}}\,dt = \lim_{n\to\infty}\left(\begin{array}{l}\text{area under } z=\frac{1}{1+t^{1/n}}\\ \text{from } t=0 \text{ to } t=1\end{array}\right)$

$= \frac{1}{2}$.