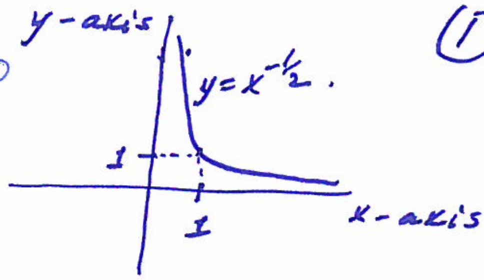
Improper Integrals: Additional stuff 2

12.04.2010



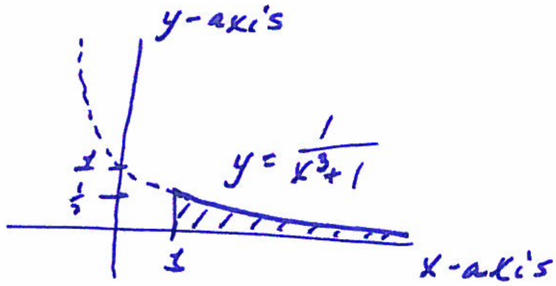
Example $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx = \int_0^\infty x^{-\frac{1}{2}}\,dx$

$$= \int_0^1 x^{-\frac{1}{2}}\,dx + \int_1^\infty x^{-\frac{1}{2}}\,dx$$

Since

$$\int_1^\infty x^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\,dx = \lim_{b\to\infty}\left(2x^{\frac{1}{2}}\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(2b^{\frac{1}{2}} - 2\cdot 1^{\frac{1}{2}}\right) = \text{diverges.}$$



Example $\displaystyle\int_1^\infty \frac{1}{x^3+1}\,dx$

$$\le \int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b x^{-3}\,dx$$

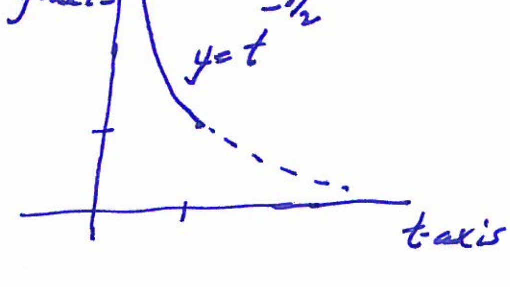
$$= \lim_{b\to\infty}\left(\frac{-1}{2}x^{-2}\right)\Big|_{x=1}^{x=b} = \lim_{b\to\infty}\left(-\frac{1}{2}b^{-2} - \left(-\frac{1}{2}1^{-2}\right)\right)$$

$$= \lim_{b\to\infty}\left(\frac{-1}{2b^2} + \frac{1}{2\cdot 1^2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

Example $\displaystyle\int_0^1 t^{-3/2}\,dt = \lim_{b\to 0}\left(\int_b^1 t^{-3/2}\,dt\right)$

$$= \lim_{b\to 0}\left(-2t^{-\frac{1}{2}}\Big|_{t=b}^{t=1}\right) = \lim_{b\to 0}\left(-2\cdot 1^{-\frac{1}{2}} - \left(-2b^{-\frac{1}{2}}\right)\right)$$

$$= \lim_{b\to 0}\left(-2 + \frac{2}{b^{\frac{1}{2}}}\right) = \text{diverges}$$

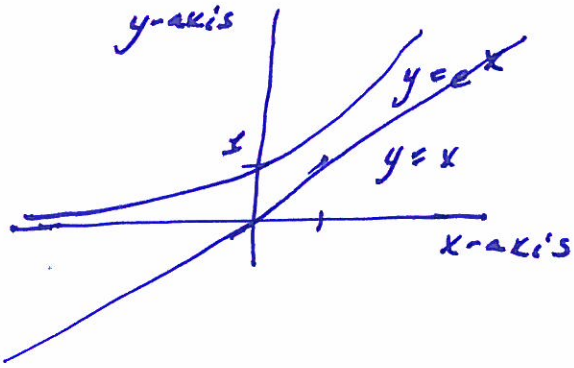


<u>Example</u> $$\int_{-\infty}^{b} e^{x-e^x}\,dx = \int_{-\infty}^{b} e^x e^{-e^x}\,dx$$

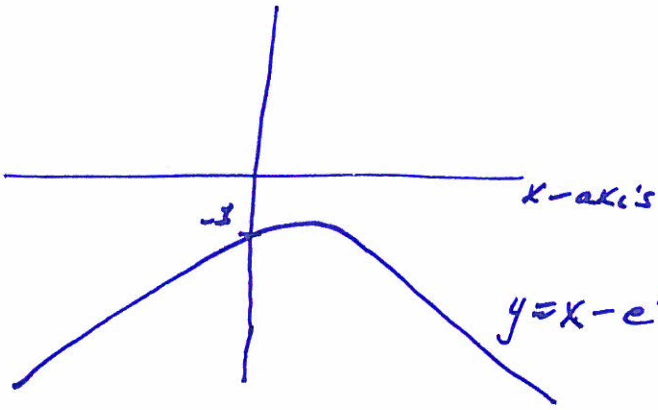
$$= -\int_{-\infty}^{b} e^{-e^x}(-e^x)\,dx = \lim_{a\to-\infty}\left(-\int_a^b e^{-e^x}(-e^x)\,dx\right)$$

$$= \lim_{a\to-\infty}\left(-\left(e^{-e^x}\right)\right)\Big|_a^b = \lim_{a\to-\infty}\left(-e^{-e^a}-\left(-e^{-e^b}\right)\right)$$

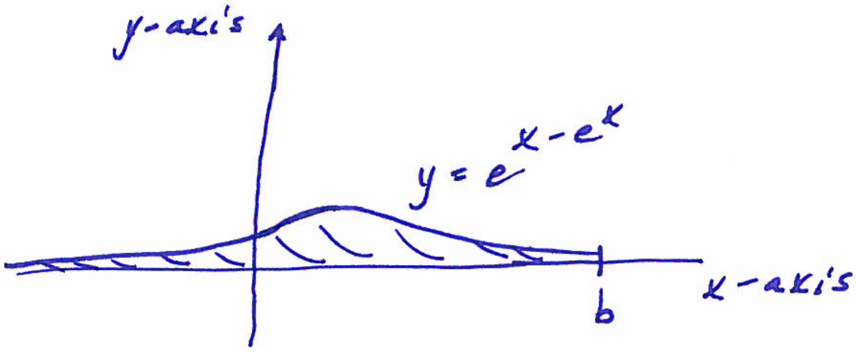
$$= -e^0 + e^{-e^b} = e^{-e^b} - 1,$$ converges.



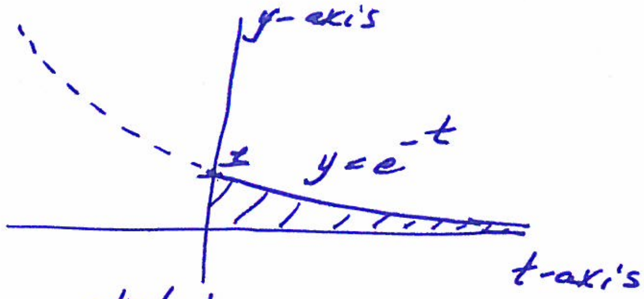
and



so that



<u>Example</u> $\int_0^\infty e^{-t}\,dt$



$$= \lim_{b\to\infty} \int_0^b e^{-t}\,dt = \lim_{b\to\infty}\left(-e^{-t}\Big|_0^{t=b} \right)$$

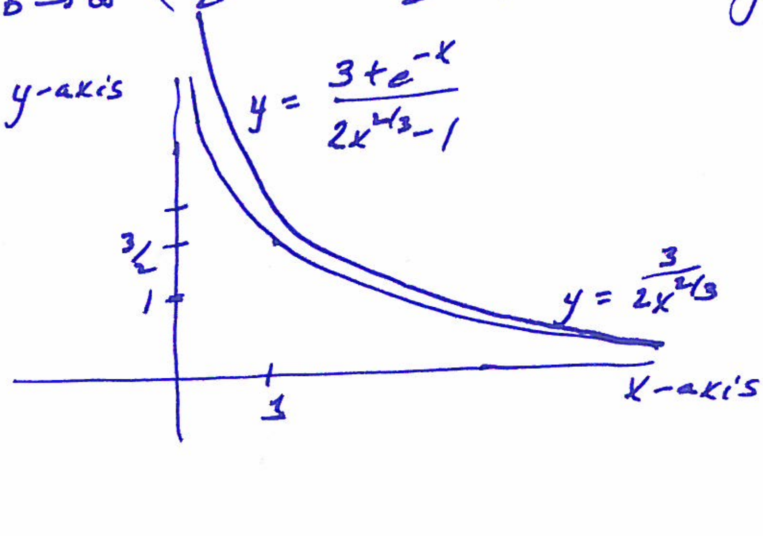
$$= \lim_{b\to\infty}\left(-e^{-b}-(-e^{-0})\right) = \lim_{b\to\infty}\left(-\frac{1}{e^b}+1\right) = 0+1=1.$$

<u>Example</u>

$$\int_1^\infty \frac{3+e^{-x}}{2x^{2/3}-1}\,dx > \int_1^\infty \frac{3}{2x^{2/3}}\,dx$$

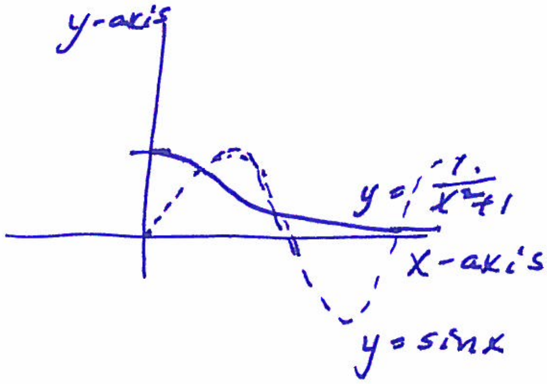
$$= \lim_{b\to\infty} \int_1^b \frac{3}{2}x^{-2/3}\,dx = \lim_{b\to\infty}\left(\frac{3}{2}\cdot 3x^{\frac{1}{3}}\Big|_{x=1}^{x=b} \right)$$

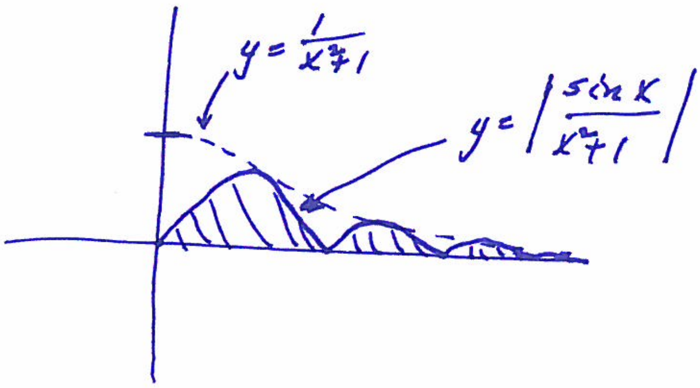
$$= \lim_{b\to\infty}\left(\frac{9}{2}b^{\frac{1}{3}} - \frac{9}{2}1^{\frac{1}{3}}\right) = \text{diverges}$$



Example

$$\int_0^\infty \left|\frac{\sin x}{x^2+1}\right| dx = \int_0^1 \left|\frac{\sin x}{x^2+1}\right| dx + \int_1^\infty \left|\frac{\sin x}{x^2+1}\right| dx$$





So $\int_0^1 \left|\frac{\sin x}{x^2+1}\right| dx$ is finite.

$$\int_1^\infty \left|\frac{\sin x}{x^2+1}\right| dx < \int_1^\infty \frac{1}{x^2+1}\, dx < \int_1^\infty \frac{1}{x^2}\, dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} \left(-x^{-1} \Big|_{x=1}^{x=b} \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right) = 0 + 1 = 1.$$

So $\int_0^\infty \left|\frac{\sin x}{x^2+1}\right| dx$ converges.

<u>Example</u> $$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt.$$

Show that $$B(u,v) = \int_0^\infty \frac{x^{u-1}}{(1+x)^{u+v}}\,dx.$$

<u>Proof.</u>

Let $t = \frac{x}{1+x}$. Then $$\frac{dt}{dx} = x(-1)(1+x)^{-2} + \frac{1}{1+x}$$

$$= \frac{-x}{(1+x)^2} + \frac{1+x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

So

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt$$

$$= \int_0^1 \left(\frac{x}{1+x}\right)^{u-1}\left(1 - \frac{x}{1+x}\right)^{v-1}\frac{dt}{dx}\,dx$$

$$= \int_0^1 \left(\frac{x}{1+x}\right)^{u-1}\left(\frac{1+x-x}{1+x}\right)^{v-1}\frac{1}{(1+x)^2}\,dx$$

$$= \int_0^1 \frac{x^{u-1}\cdot 1^{v-1}}{(1+x)^{u+v-2}(1+x)^2}\,dx = \int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}}\,dx.$$