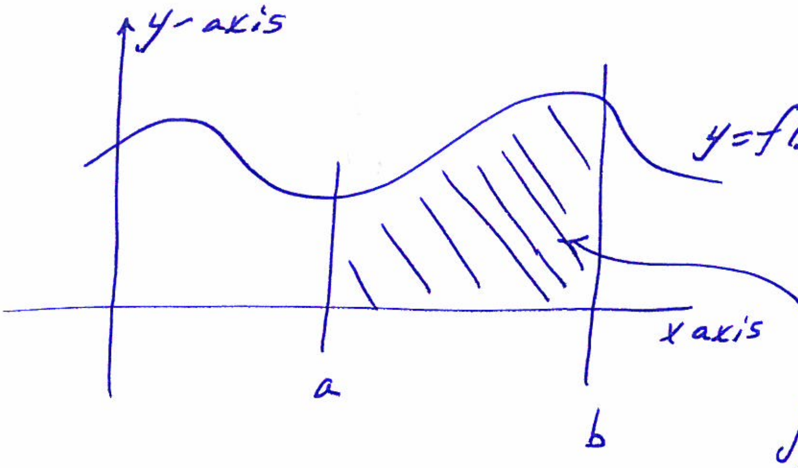
620-295 Real Analysis with applications

Improper integrals

Idea:

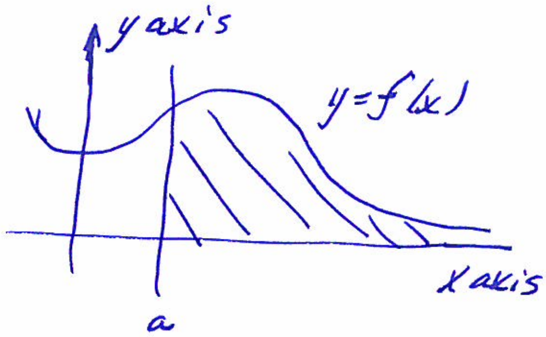
$$\int_a^b f(x)\,dx = \left(\begin{array}{l}\text{area under } f(x) \\ \text{from } x=a \text{ to } x=b.\end{array}\right)$$



This area is $\int_a^b f(x)\,dx$

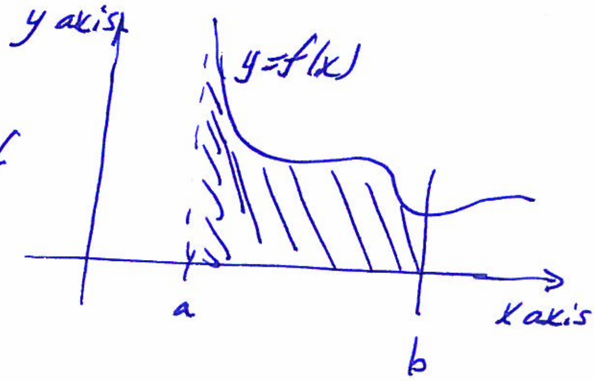
Improper integrals: Definitions

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$



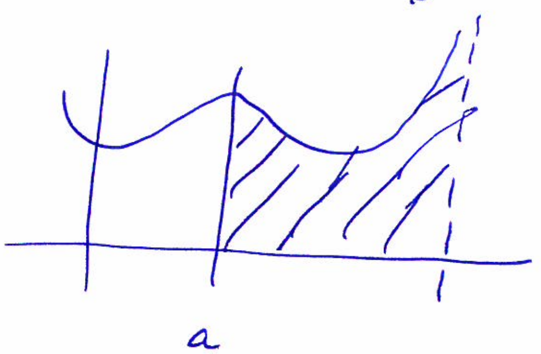
and

$$\int_a^b f(x)\,dx = \lim_{l\to a} \int_l^b f(x)\,dx \quad \text{if}$$

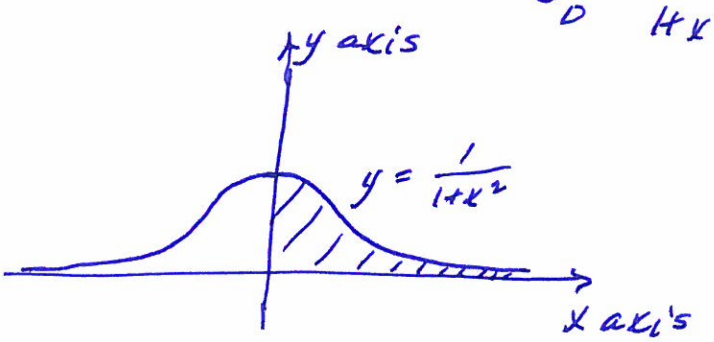


or

$$\int_a^b f(x)\,dx = \lim_{l\to b} \int_a^l f(x)\,dx \quad \text{if}$$



<u>Example</u> Evaluate $\int_0^\infty \frac{dx}{1+x^2}$,



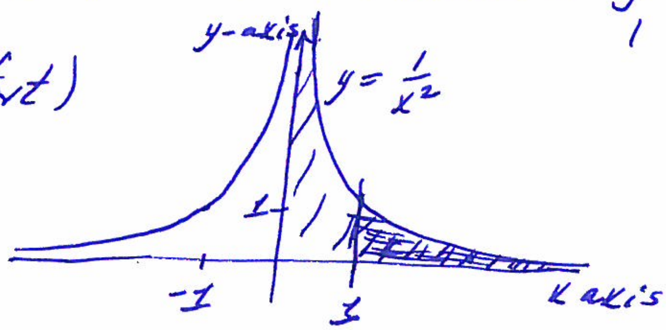
$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \left(\tan^{-1}x \Big|_{x=0}^{x=t} \right) = \lim_{t\to\infty} \left(\tan^{-1}t - \tan^{-1}0 \right)$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - 0 \right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

<u>Example</u> Let $p \in \mathbb{R}$, $p > 1$. Evaluate $\int_1^\infty \frac{1}{x^p}\,dx$.

(Think $p = 2$ for comfort)

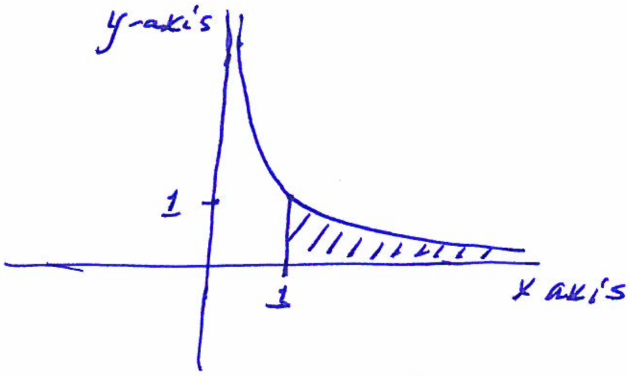


$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left(\frac{x^{-p+1}}{-p+1} \Big|_{x=1}^{x=t} \right)$$

$$= \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \lim_{t\to\infty} \left(\left(\frac{1}{1-p}\right) \frac{1}{t^{p-1}} - \frac{1}{1-p} \right)$$

$$= 0 - \frac{1}{1-p} = \frac{1}{p-1}.$$

<u>Example</u> Let $p=1$. Evaluate $\int_1^\infty \frac{1}{x^p}\,dx$

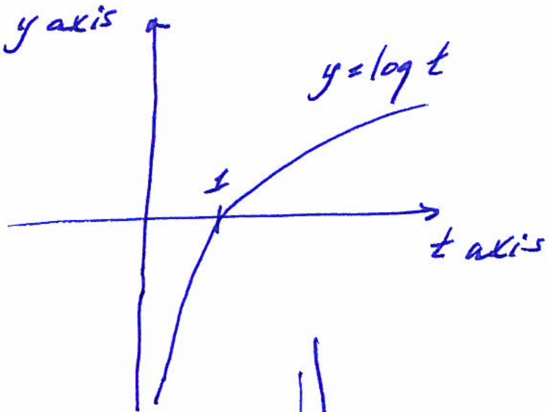


$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x}\,dx$$

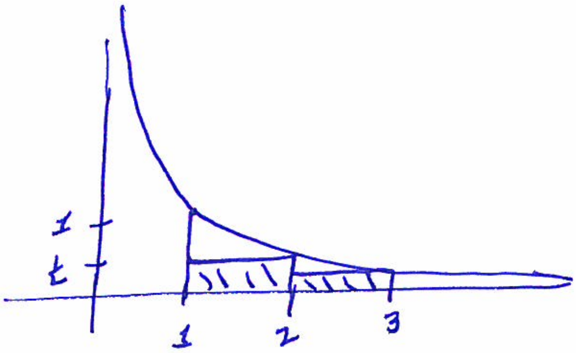
$$= \lim_{t\to\infty} \int_1^t x^{-1}\,dx = \lim_{t\to\infty} \left(\log x \Big|_{x=1}^{x=t} \right)$$

$$= \lim_{t\to\infty} (\log t - \log 1) = \lim_{t\to\infty} \log t \quad \text{diverges.}$$

since



Alternatively,



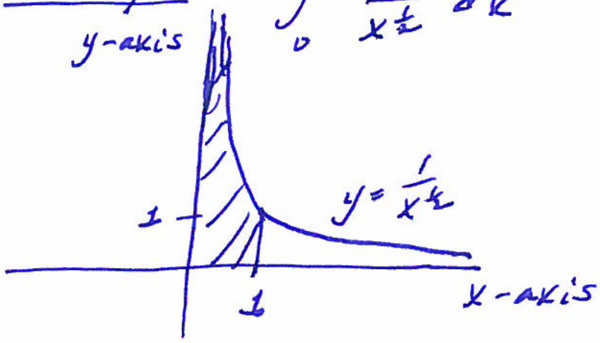
$$\int_1^\infty \frac{1}{x}\,dx > \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$$

$$> \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \quad \text{diverges.}$$

Evaluate

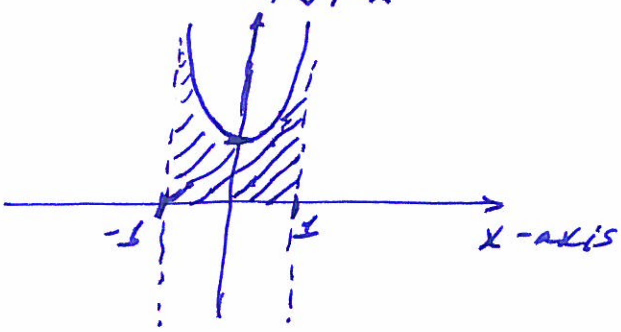
<u>Example</u> $\displaystyle\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx$



$$\int_0^1 x^{-\frac{1}{2}}\,dx = \lim_{t\to 0}\int_t^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 0}\left(2x^{\frac{1}{2}}\Big|_{x=t}^{x=1}\right) = \lim_{t\to 0}\left(2-2t^{\frac{1}{2}}\right) = 2.$$

<u>Example</u> $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

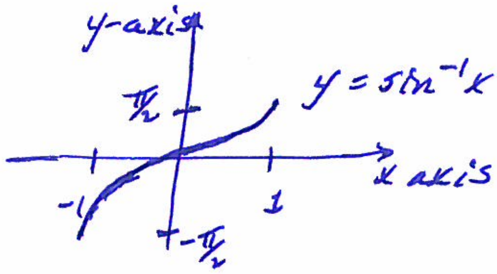


$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{t\to 1} 2\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1}\left(2\left(\sin^{-1}x\right)\Big|_{x=0}^{x=t}\right)$$

$$= \lim_{t\to 1}\left(2\sin^{-1}t - 2\sin^{-1}0\right)$$

$$= 2\,\frac{\pi}{2} - 2\cdot 0 = \pi.$$



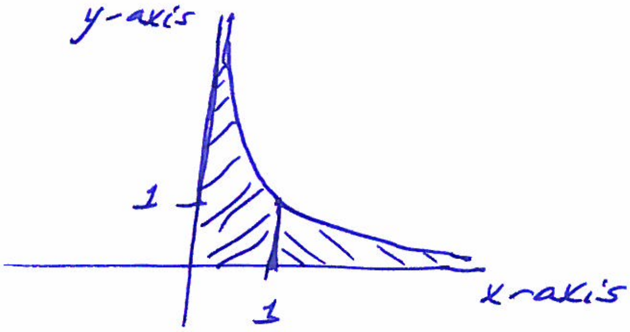
<u>Example</u> Let $p \in \mathbb{R}_{\geq 0}$ with $p<1$.

Evaluate $\int_0^1 \frac{1}{x^p}\,dx$. (Think $p=\frac{1}{2}$ for comfort.)

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0}\int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to 0}\left(\frac{x^{-p+1}}{-p+1}\Big|_{x=t}^{x=1}\right)$$

$$= \lim_{t\to 0}\left(\frac{1^{-p+1}}{-p+1} - \frac{t^{-p+1}}{-p+1}\right) = \lim_{t\to 0}\left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p}\right) = \frac{1}{1-p} - 0 = \frac{1}{1-p}.$$

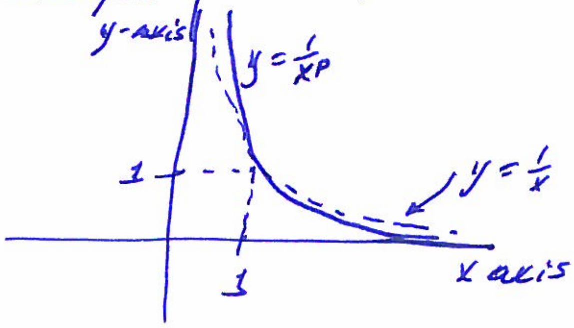
<u>Example</u> Evaluate $\int_0^1 \frac{1}{x}\,dx$



From the picture,

$\int_0^1 \frac{1}{x}\,dx = 1 + \int_1^\infty \frac{1}{x}\,dx$ diverges (from earlier computation)

<u>Example</u> Let $p \in \mathbb{R}$ with $p>1$. Evaluate $\int_0^1 \frac{1}{x^p}\,dx$



Between $x=0$ and $x=1$

$\frac{1}{x^p} \geq \frac{1}{x}$.

So $\int_0^1 \frac{1}{x^p}\,dx \geq \int_0^1 \frac{1}{x}\,dx$ which diverges.

So $\int_0^1 \frac{1}{x^p}\,dx$ diverges.