Problem Set: Improper integrals
620-205 Semester I 2010

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1. Improper Integrals: Rational functions

For each of the following integrals:

(a) graph the integrand,
(b) determine if the integral converges, and
(c) evaluate the integral as appropriate.

(1) $$\int_0^1 \frac{1}{x}\,dx$$

(2) $$\int_0^1 \frac{1}{x^3}\,dx$$

(3) $$\int_{-1}^1 \left(1-x^2\right)^n \, dx$$

(4) $$\lim_{n\to\infty} \int_0^1 \frac{ny^{n-1}}{1+y} \, dy$$

(5) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

(6) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

(7) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(8) Show that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p \in \mathbb{R}$ and $p > 1$.

(9) Show that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ diverges if $p \in \mathbb{R}$ and $p \leq 1$.

(10) $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$

(11) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(12) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$

(13) $\displaystyle\int_1^\infty \frac{1}{x^{1.001}}\,dx$

(14) $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

(15) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

(16) $\displaystyle\int_0^1 \frac{1}{x^{0.999}}\,dx$

(17) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

(18) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

(19) $\displaystyle\int_1^\infty \frac{1}{x^3+1}\,dx$

(20) $\displaystyle\int_0^\infty \frac{1}{x^3}\,dx$

(21) $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx$

(22) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

(23) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/5}}\,dx$

(24) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx$

(25) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+x^4}}\,dx$

(26) $\displaystyle\int_1^5 \frac{4x}{\sqrt{x^2-1}}\,dx$

(27) $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$

(28) $\displaystyle\int_1^\infty \frac{x^2}{(x-2)(x^{11}+2)^{1/4}}\,dx$

(29) Let $a, b \in \mathbb{R}$ with $a < b$. Analyse $\int_a^b x^{-\frac{1}{2}} dx$.

(30) Let $a, b \in \mathbb{R}$ with $a < b$. Analyse $\int_a^b \frac{1}{x} dx$.

(31) Let $\alpha \in \mathbb{R}$. Analyse $\int_0^1 x^\alpha dx$.

(32) Let $a \in \mathbb{R}$. Analyse $\int_{-a}^a \frac{1}{x} dx$.

(33) $\int_0^\infty \frac{1}{x+1} dx$

(34) Let $m \in \mathbb{R}_{\leq 0}$. Analyse $\int_0^\infty t^{-m} dt$.

(35) $\int_{-\infty}^\infty \frac{1}{1+t^2} dt$

(36) $\int_1^\infty t^{-1/2} dt$

(37) $\int_1^\infty t^{-3/2} dt$

(38) $\int_0^1 t^{-1/2} dt$

(39) $\int_0^1 t^{-3/2} dt$

(40) $\int_4^\infty \frac{1}{t^2} dt$

(41) $\displaystyle\int_{2.735}^{\infty} \frac{1}{t^{1/3}}\,dt$

(42) Let $p \in \mathbb{R}_{>0}$. Analyse $\displaystyle\int_{1}^{\infty} \frac{1}{t^{p}}\,dt$.

(43) $\displaystyle\int_{0}^{1} \frac{1}{t-\frac{1}{2}}\,dt$

2. Improper Integrals: Exponential functions

For each of the following integrals:

(a) graph the integrand,
(b) determine if the integral converges, and
(c) evaluate the integral as appropriate.

(1) $\displaystyle\int_{3}^{\infty} e^{-3x}\,dx$

(2) $\displaystyle\int_{0}^{\infty} e^{-x^2}\,dx$

(3) $\displaystyle\int_{0}^{\infty} x^3 e^{-x^2}\,dx$

(4) $\displaystyle\int_{-\infty}^{b} e^{x-e^x}\,dx$

(5) $\displaystyle\int_{-\infty}^{\infty} e^{x-e^x}\,dx$

(6) $\displaystyle\int_{1}^{\infty} e^{-x^2}\,dx$

(7) $\displaystyle\int_{0}^{\infty} e^{-x}\cos x \quad dx$

(8) $\displaystyle\int_0^\infty \frac{1}{1+e^x}\,dx$

(9) $\displaystyle\int_0^\infty e^{-t}\,dt$

(10) $\displaystyle\int_0^\infty e^{-x}\,dx$

(11) $\displaystyle\int_{-\infty}^\infty e^{-t^2}\,dt$

(12) $\displaystyle\int_3^\infty \frac{1}{\sqrt{x+e^{2x}}}\,dx$

(13) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,dx$

(14) $\displaystyle\int_1^\infty \frac{3+e^{-x}}{2x^{2/3}-1}\,dx$

3. Improper Integrals: Special functions

For each of the following integrals:

(a) graph the integrand,
(b) determine if the integral converges, and
(c) evaluate the integral as appropriate.

(1) $\displaystyle\int_0^\infty \cos x \quad dx.$

(2) $\displaystyle\int_0^{\pi/2} \tan x \quad dx$

(3) $\displaystyle\int_0^1 \log x \quad dx$

(4) $$\int_1^\infty \frac{(\log x)^4}{1+x^2}\,dx$$

(5) $$\int_{-\pi}^{\pi} \frac{\sin x}{|x|^\beta} \quad dx$$

(6) $$\int_0^1 t^{1/2} e^{e^t} \quad dt$$

(7) $$\int_0^{100} e^{\lfloor t \rfloor} dt$$

(8) $$\int_3^\infty \frac{\log x}{\sqrt{x^{3/2}+1}}\,dx$$

(9) $$\int_0^\infty \left| \frac{\sin x}{x^2+1} \right| \quad dx$$

(10) $$\int_0^\infty \frac{\sin x}{x} \quad dx$$

(11) $$\int_0^\infty x^{\alpha-1} \cos x \quad dx$$

(12) $$\int_0^\infty x^{\alpha-1} \sin x \quad dx$$

(13) $$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} \quad dt$$

(14) Let $B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} \quad dt$. Show that $B(u, v) = \int_0^\infty \frac{x^{u-1}}{(1+x)^{u+v}} \quad dx$ by setting $t = \frac{x}{1+x}$.

4. Improper Integrals: Analysis and applications

(1) Let $A,\ r \in \mathbb{R}$. Analyse $\int_0^\infty Ae^{-rt} \quad dt$.

(2) Show that if $z = \sqrt{\frac{4+x}{1-x}}$ then $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} dx = \int_0^\infty \frac{4z^2}{(z^2+1)^2} dz$. The improper integral on the left is an improper integral of the first kind and the improper integral on the right is an improper integral of the second kind.

(3) Show that $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} dx = \pi$.

(4) Let $n \in \mathbb{Z}_{\geq 0}$. Show that $n! = \int_0^\infty e^{-t} t^n dt$.

(5) Define $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$. Show that $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$.

(6) Let $f(x) = \begin{cases} e^x, & \text{if } x \in \mathbb{R}_{\geq 0} \text{ and } x \in \mathbb{Q}, \\ e^{-x}, & \text{if } x \in \mathbb{R}_{\geq 0} \text{ and } x \notin \mathbb{Q}. \end{cases}$

Show that $\int_0^\infty \left| f(x) \right| \quad dx$ exists but $\int_0^\infty f(x) \quad dx$ doesn't.

(7) Let $B(m+1) = \int_0^1 x^{m-1}(1-x)^{-1/2} \quad dx$.

(a) Explain why $B(m)$ is improper for all values of m.

(b) Find the value of $B(1)$.

(c) Show that $B(m+1) = \frac{2m}{2m+1} B(m)$.

(d) Find the values of $B(2), B(3)$ and $B(4)$.

(8) Let $\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} \quad dx.$

(a) Explain why $\Gamma(p)$ is improper for all values of p.

(b) Evaluate $\Gamma(1)$ and $\Gamma(2)$.

(c) Show that $\Gamma(p+1) = p\Gamma(p)$.

(d) Find the values of $\Gamma(3)$, $\Gamma(4)$ and $\Gamma(5)$.

5. References

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[TF] Thomas and Finney, *Calculus and Analytic Geometry*, Fifth Edition, Addison-Wesley 1979.