Assignment 1 -- Expressions and Graphing
620-295 Semester I 2010

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1. Some definitions. Define the following:

(1) $\tan x$ (2) $\cosh x$ (3) $\tanh x$ (4) $\sin^{-1} x$

(5) $\arctan x$ (6) $\csc^{-1} x$ (7) $\operatorname{arccosh} x$ (8) $\operatorname{sech}^{-1} x$

2. Basic properties. Prove the following basic statements:

(1) $\ln x$ is the inverse function to e^x. (2) $\ln(xy) = \ln x + \ln y$.

(3) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$. (4) $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

(5) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$. (6) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(7) $\operatorname{arccosh} x = \log\left(x + \sqrt{x^2-1}\right)$.

3. Trig Identities. Review your trig by proving the following equalities:

(1) $\sin^2 A \cot^2 A = (1 - \sin A)(1 + \sin A)$. (2) $\dfrac{\sec A - 1}{\sec A + 1} + \dfrac{\cos A - 1}{\cos A + 1} = 0$.

(3) $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1$. (4) $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

(5) $\cos(\pi/4 - x) - \cos(\pi/4 + x) = \sqrt{2}\sin x$. (6) $\cot(x/2) = \dfrac{1 + \cos x}{\sin x}$.

(7) $\csc A \sec A = 2\csc 2A$. (8) $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha$.

(9) $\cos^2\theta = \dfrac{\cot^2\theta}{1 + \cot^2\theta}$. (10) $\dfrac{2\tan^2 A}{1 + \tan^2 A} = 1 - \cos 2A$.

4. Inverse expressions. Review your inverse trig functions by proving the following equalities:

(1) $\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$ (2) $\sin^{-1}(-x) = -\sin^{-1} x$

(3) $\operatorname{arcsinh}\left(\dfrac{x}{\sqrt{1 - x^2}}\right) = \operatorname{arctanh} x$

5. Derivatives. Review the basics of derivatives with the following:

Let $D : \mathbb{Q}[x] \to \mathbb{Q}[x]$ be a function such that

(D1) If $f, g \in \mathbb{Q}[x]$ then $D(f+g) = D(f) + D(g)$,
(D2) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$ then $D(cf) = cD(f)$,
(D3) If $f, g \in \mathbb{Q}[x]$ then $D(fg) = fD(g) + D(f)g$ and
(D4) $D(x) = 1$.

(1) Prove that if $n \in \mathbb{Z}_{>0}$ then $D(x^n) = nx^{n-1}$.

(2) Prove that if $f = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 +$ then $c_k = \frac{1}{k!}(D^k f)|_{x=0}$.

6. Series expansions. Find a series expansion for each of the following:

(1) $\cos x$ (2) $\dfrac{1}{1+x^2}$ (3) $\cosh x$ (4) $\sinh x$

(5) $\dfrac{1}{1+x}$ (6) $x \sin 3x$ (7) $\int e^{x^3}\,dx$ (8) $\int_0^t \sin\left(x^2\right)\,dx$

7. Alternate formulations of series expansions.

(1) Find the Taylor series for e^x at the point $a = -3$.

(2) Find an alternate expression for the series $\sum_{n=2}^{\infty} n(n-1)x^n$.

(3) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n2^{n+1}}$.

8. Some basic graphs. Graph the following:

(1) $f(x) = x^2$ (2) $f(x) = x^{100}$ (3) $f(x) = x^{-100}$ (4) $f(x) = \cot x$

(5) $f(x) = x^{1/4}$ (6) $f(x) = x^{-1/4}$ (7) $f(x) = \operatorname{arccot} x$

9. Additional graphing. Make sure your graphing skills are at a level where they won't hamper performance later in the course by graphing the following:

(1) $f(x) = \begin{cases} 2+x, & \text{if } x > 0, \\ 2-x, & \text{if } x \le 0. \end{cases}$ (2) $f(x) = x^3$ (3) $f(x) = 2x^3 + x^2 + 20x$

(4) $f(x) = x^3(x-2)^2$

(5) $f(x) = \sqrt{x+2}$

(6) $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

(7)
$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ \text{-}1, & \text{if } x < 0. \end{cases}$$

(8) $f(x) = x^{2/3}(6-x)^{1/3}$

(9) $f(x) = \sin 2x - x$

(10) $y = e^{-x}$

10. General ellipses and hyperbolas. Recall how to graph ellipses and hyperbolas by graphing the following:

(1) Graph $f(x) = y$, where $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$, and h, k and r are constants.

(2) Graph $f(x) = y$, where $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and a and b are constants.

11. Using graphs to view sequences. Graph the following sequences:

(1) $a_n = n!$.

(2) $a_n = \dfrac{n}{n(n+1)}$

(3) $a_n = \dfrac{n}{2n+1}$

(4) $a_n = \dfrac{i^n}{n^2}$

(5) $a_n = \dfrac{(n!)^2 5^n}{(2n)!}$.

(6) $a_n = \sqrt{n}$.

12. Sequences in recursive form. Graph the following sequences:

(1) Let $a \in \mathbb{R}$ with $a > 0$. Fix a positive real number x_1 and let $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$.

(2) Graph the sequence given by $a_1 = 0$, $a_{2k} = \frac{1}{2}a_{2k+1}$, and $a_{2k+1} = \frac{1}{2} + a_{2k}$.

13. Detecting continuity from a graph.

(1) Let $k \in \mathbb{R}$. Graph

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0. \end{cases}$$

For which values of k is the function continuous?

(2) Graph

$$f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \le x \le 1, \\ 2 - x, & \text{if } x > 1. \end{cases}$$

For which values of x is the function continuous?

(3) Graph

$$f(x) = \begin{cases} \sin x, & \text{if } x < 0, \\ x, & \text{if } x \ge 0. \end{cases}$$

For which values of x is the function continuous?

(4) Graph

$$f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \ne 1, \\ 4, & \text{if } x = 1. \end{cases}$$

For which values of x is the function continuous?

14. Detecting existence of limits from a graph.

(1) Graph $y = \ln(x)$ and explain why $\lim\limits_{x \to -1} \ln(x)$ does not exist.

(2) Graph $y = 2^{1/(1-x)}$ and explain why $\lim\limits_{x \to 1} 2^{1/(1-x)}$ does not exist.