The Ratio test

Example Does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge?

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} = 1 + \frac{2}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$$

Look at
$$\left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \left| \frac{(n+1)!\, n^n}{n!\, (n+1)^{n+1}} \right| = \left| \frac{(n+1)\, n^n}{(n+1)^{n+1}} \right|$$

$$= \frac{n^n}{(n+1)^{\cancel{n+1}}} = \left(\frac{n}{n+1} \right)^n = \frac{1}{(1+\frac{1}{n})^n}$$

$< \frac{1}{2}$ if n is large enough.

In fact $\frac{1}{(1+\frac{1}{1})^1} = \frac{1}{2}$ and $\frac{1}{(1+\frac{1}{2})^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} < \frac{1}{2}$.

So
$$\sum_{n=1}^{\infty} \frac{n!}{n^n} = 1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$$

$$= 1 + \frac{2!}{2^2} + \frac{2!}{2^2}\left(\frac{\frac{3!}{3^3}}{\frac{2!}{2^2}} \right) + \frac{2!}{2^2}\left(\frac{\frac{3!}{3^3}}{\frac{2!}{2^2}} \right)\left(\frac{\frac{4!}{4^4}}{\frac{3!}{3^3}} \right) + \cdots$$

$$< 1 + \frac{2!}{2^2} + \frac{2!}{2^2}\cdot\frac{1}{2} + \frac{2!}{2^2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \cdots$$

$$= 1 + \frac{2!}{2^2}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right)$$

$$= 1 + \frac{2!}{2^2}\left(\frac{1}{1-\frac{1}{2}} \right) = 1 + \frac{2}{4}\cdot 2 = 1 + 1 = 2.$$

So $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

<u>Theorem</u> Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$.

(a) Assume $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = a$ exists and $a<1$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

(b) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ exists and $a>1$.

Then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>

(a) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ exists and $a<1$

Let $\varepsilon \in \mathbb{R}_{>0}$ so that $a+\varepsilon<1$.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ there exists $N \in \mathbb{Z}_{>0}$

with $\frac{a_{n+1}}{a_n} < a+\varepsilon$ if $n \in \mathbb{Z}_{>0}$ with $n>N$.

Then
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_N + a_{N+1} + a_{N+2} + \cdots$$
$$= a_1 + \cdots + a_N + a_{N+1} + a_{N+1}\left(\frac{a_{N+2}}{a_{N+1}}\right) + a_{N+1}\left(\frac{a_{N+2}}{a_{N+1}}\right)\left(\frac{a_{N+3}}{a_{N+2}}\right) + \cdots$$
$$\leq a_1 + \cdots + a_N + a_{N+1} + a_{N+1}(a+\varepsilon) + a_{N+1}(a+\varepsilon)^2 + \cdots$$

$$= a_1 + \cdots + a_N + a_{N+1}\left(1 + (a+\varepsilon) + (a+\varepsilon)^2 + (a+\varepsilon)^3 + \cdots\right)$$

$$= a_1 + \cdots + a_N + a_{N+1}\left(\frac{1}{1-(a+\varepsilon)}\right).$$

So $\sum_{n=1}^{\infty} a_n$ converges.

(b) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and $a > 1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $a - \varepsilon > 1$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$

then $\frac{a_{n+1}}{a_n} > a - \varepsilon$.

Then

$$\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots$$

$$= a_1 + \cdots + a_N + a_{N+1}\left(1 + \frac{a_{N+2}}{a_{N+1}} + \left(\frac{a_{N+2}}{a_{N+1}}\right)\left(\frac{a_{N+3}}{a_{N+2}}\right) + \cdots\right)$$

$$> a_1 + \cdots + a_N + a_{N+1}\left(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots\right)$$

$$> a_1 + \cdots + a_N + a_{N+1}\left(1 + 1 + 1 + 1 + \cdots\right)$$

So $\sum_{n=1}^{\infty} a_n$ diverges. //

Radius of convergence

Example Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n}}$

i.e. for which x does the series converge?

To analyse $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{y^n}{n^{1/3}}$, if $y = 2x-1$.

Look at the ratios: $\left| \frac{\frac{y^{n+1}}{(n+1)^{1/3}}}{\frac{y^n}{n^{1/3}}} \right| = \frac{|y| n^{1/3}}{(n+1)^{1/3}} = \frac{|y|}{(1+\frac{1}{n})^{1/3}}$

As $n \to \infty$,

$$\lim_{n\to\infty} \frac{|y|}{(1+\frac{1}{n})^{1/3}} = \frac{|y|}{(1+0)^{1/3}} = |y|.$$

So, if $|y|<1$ then $\sum_{n=1}^{\infty} \left|\frac{y^n}{n^{1/3}}\right|$ converges

and if $|y|>1$ then $\sum_{n=1}^{\infty} \left|\frac{y^n}{n^{1/3}}\right|$ diverges.

If $|y|=1$ then $\sum_{n=1}^{\infty} \left|\frac{y^n}{n^{1/3}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} > \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

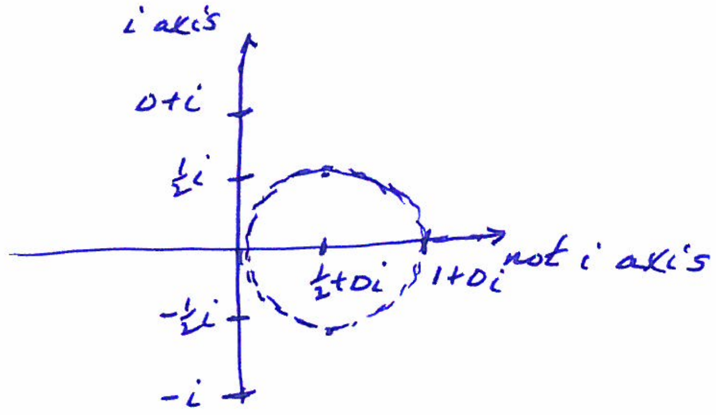
So, we can be sure that

if $|y|<1$ then $\sum_{n=1}^{\infty} \frac{y^n}{n^{1/3}}$ converges.

So if $|2x-1|<1$ then $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{1/3}}$ converges

So, if $|x-\frac{1}{2}|<\frac{1}{2}$ then $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{1/3}}$ converges

So, if x is inside the circle



then $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{1/3}}$ converges.

__Theorem__ Let $r, s \in \mathbb{C}$.

Assume $\displaystyle\sum_{n=0}^{\infty} a_n s^n$ converges.

If $|r| < |s|$ then $\displaystyle\sum_{n=0}^{\infty} a_n |r|^n$ converges.

__Proof__ Since $\displaystyle\sum_{n=0}^{\infty} a_n s^n$ converges $\displaystyle\lim_{n\to\infty} |a_n s^n| = 0$.

Let $\varepsilon \in \mathbb{R}_{>0}$. Then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $|a_n s^n| < \varepsilon$.

~~Let~~ Then

$$\sum_{n=0}^{\infty} |a_n r^n| = |a_0| + |a_1 r| + \cdots + |a_N r^N| + |a_{N+1} r^{N+1}| + \cdots$$

$$= |a_0| + \cdots + |a_N r^N| + |a_{N+1} s^{N+1}| \left|\frac{r^{N+1}}{s^{N+1}}\right| + |a_{N+2} s^{N+2}| \left|\frac{r^{N+2}}{s^{N+2}}\right| + \cdots$$

$$< |a_0| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| + \varepsilon \left|\frac{r^{N+2}}{s^{N+2}}\right| + \varepsilon \left|\frac{r^{N+3}}{s^{N+3}}\right| + \cdots$$

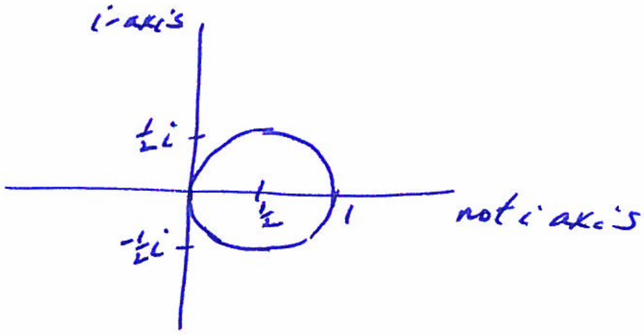
$$= |a_0| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| \left(1 + \left|\frac{r}{s}\right| + \left|\frac{r^2}{s^2}\right| + \cdots\right)$$

$$= |a_0| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| \left(\frac{1}{1 - \frac{|r|}{|s|}}\right)$$

So $\displaystyle\sum_{n=0}^{\infty} |a_n||r^n|$ converges. So $\displaystyle\sum_{n=0}^{\infty} a_n r^n$ converges. //

It follows that,

if x is outside the circle



then $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{1/3}}$ diverges.

What ~~is~~ If x is on the circle?