Lectures 13 and 14, 29 and 31 March 2010

Conditional convergence and alternating series

Theorem Let a_n be a sequence in

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Favourite example Does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converge?

$$\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

then $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

and $\log(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

So if $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ converges it is equal to $\log(1+1) = \log 2$.

A quick experiment indicates that it does converge.

However

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges } \left(\begin{matrix}\text{harmonic}\\ \text{series}\end{matrix}\right).$$

So $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is conditionally convergent and not absolutely convergent.

<u>Liebniz's</u> <u>Theorem</u> If (a_n) is a sequence in $\mathbb{R}$ such that

(a) $a_n \in \mathbb{R}_{\geq 0}$,

(b) if $n \in \mathbb{Z}_{>0}$ then $a_n \geq a_{n+1}$,

(c) $\lim_{n\to\infty} a_n = 0$

then $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ converges.

<u>Proof</u> Assume (a_n) is a sequence in $\mathbb{R}$ and $a_n \in \mathbb{R}_{\geq 0}$, and if $n \in \mathbb{Z}_{>0}$ then $a_n \geq a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$.

To show: $\sum_{n=1}^{\infty}(-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.

Let

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

Then $s_{2m} \leq s_{2(m+1)}$

Since $s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$,

then $s_{2m} \le a_1$

So the sequence $(s_2, s_4, s_6, \ldots)$ is increasing and bounded above. So $\lim\limits_{n\to\infty} s_{2m}$ exists.

Let $l = \lim\limits_{m\to\infty} s_{2m}$.

Since $s_{2m+1} = s_{2m} + a_{2m+1}$ then

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1}$$

$$= l + 0 = l.$$

So $\lim\limits_{m\to\infty} s_m = l$. //.

<u>Example</u> $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\log n}{\sqrt{n}}$.

Since $\dfrac{d}{dx}\left(\dfrac{\log x}{\sqrt{x}}\right) = \dfrac{d}{dx}\left(x^{-\frac{1}{2}} \log x\right)$

$$= x^{-\frac{1}{2}} \frac{1}{x} + \frac{-1}{2} x^{-3/2} \log x = x^{-3/2}\left(1 - \tfrac{1}{2}\log x\right)$$

is less than 0 for $x > e^2$,

the sequence $\dfrac{\log n}{\sqrt{n}}$ is decreasing for $x > 9$.

Since $\lim\limits_{n\to\infty} \dfrac{\log n}{n^{1/2}} = \lim\limits_{n\to\infty} e^{-\frac{1}{2}\log n} \log n = \lim\limits_{y\to\infty} e^{-\frac{1}{2}y} y = 0$

Leibniz theorem gives that $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\log n}{\sqrt{n}}$ converges.

Thinking about rearrangements

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots = \log 2.$$

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots = -\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$
$$= -\frac{1}{2}(\text{VERY VERY LARGE})$$

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots > \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots = \text{VERY VERY LARGE}$$

Pick a number $\ell \in \mathbb{R}_{>0}$

If we take

$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{299}$ just until we get larger than ℓ,

then add

$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{70}$ just until we get smaller than ℓ,

then add

$\frac{1}{301}+\frac{1}{303}+\cdots$ just until we get larger than ℓ again,

then add

$-\frac{1}{72}-\frac{1}{74}-\cdots$ just until we get smaller than ℓ again

$\vdots$

This process will create a series that converges to ℓ.

<u>Theorem</u> Assume that a_n is a sequence in $\mathbb{R}$ or $\mathbb{C}$.
If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

<u>Proof</u> Let $A_n = |a_1| + |a_2| + \cdots + |a_n|$ and

$$s_n = a_1 + a_2 + \cdots + a_n$$

Since $\sum_{n=1}^{\infty} |a_n| = (A_1, A_2, A_3, \ldots)$ converges

the sequence A_n is Cauchy.

Since

$$|s_m - s_n| = |a_{n+1} + \cdots + a_m|$$

$$\leq |a_{m+1}| + \cdots + |a_n| = |A_m - A_n|,$$

the sequence s_n is Cauchy.

Since Cauchy sequences converge in $\mathbb{R}$ or $\mathbb{C}$
(or any complete metric space)

$\sum_{n=1}^{\infty} a_n = (s_n)$ converges. //