Lecture ~~11~~ 12, 26 March 2010

Definition

Let (a_n) be a sequence in $\mathbb{R}$ or $\mathbb{C}$.

The series $\sum_{n=1}^{\infty} a_n$ is the sequence $(s_1, s_2, s_3, \ldots)$,

where $s_k = a_1 + a_2 + \cdots + a_k$.

The series $\sum_{n=1}^{\infty} a_n$ converges to L if the sequence $(s_1, s_2, s_3, \ldots)$ converges to L.

Write $\sum_{n=1}^{\infty} a_n = L$ if $\sum_{n=1}^{\infty} a_n$ converges to L.

Harmonic series and the Riemann zeta function

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}} + \underbrace{\frac{1}{8^2}} + \cdots$$

$$< 1 + \frac{2}{2^2} + \frac{4}{4^2} + \frac{8}{8^2} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

In fact, according to Wolfram alpha,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

If $k > 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \underbrace{\frac{1}{2^k} + \frac{1}{3^k}} + \underbrace{\frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{6^k} + \frac{1}{7^k}} + \underbrace{\frac{1}{8^k}} + \cdots$$

$$< 1 + \frac{2}{2^k} + \frac{4}{4^k} + \frac{8}{8^k} + \cdots$$

$$= 1 + \frac{1}{2^{k-1}} + \frac{1}{4^{k-1}} + \frac{1}{8^{k-1}} + \cdots$$

$$= 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \left(\frac{1}{2^{k-1}}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{1}{2^{k-1}}} = \frac{2^{k-1}}{2^{k-1} - 1}$$

so that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges.

If $k < 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

so that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$ diverges.

<u>Theorem</u> Let $k \in \mathbb{R}_{>0}$.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges if $k > 1$ and

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^k}$ diverges if $k \le 1$.

Let $s \in \mathbb{C}$. The Riemann zeta function at s is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

<u>Example</u> $\zeta(2) = \pi^2/6$.

~~Integral tests~~

~~$> a_1 + \cdots + a_N + a_{N+1}(1+1+1+\cdots)$~~

~~So $\sum_{n=1}^{\infty} a_n$ diverges.~~

~~(c) and (d) will be on the next assignment.~~

Integral tests

The fundamental theorem of calculus (FTC) says: Let

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } \frac{dF}{dx} = f.$$

Then

$$\int_a^b f(x)\,dx = \begin{pmatrix} \text{area under } y = f(x) \text{ between} \\ x = a \text{ and } x = b \end{pmatrix}$$

if you can make good sense of what "area under $y = f(x)$ between $x = a$ and $x = b$" means

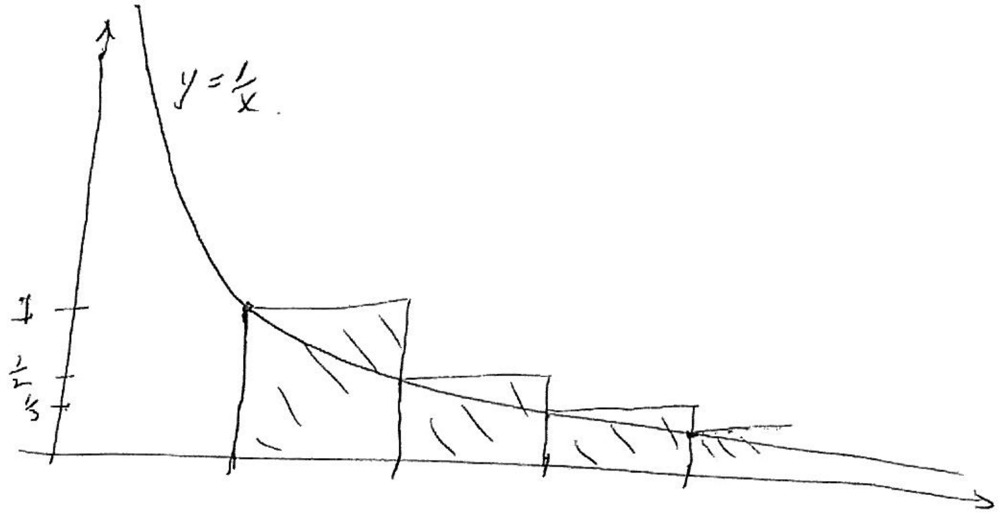
We want to use FTC to think about series.

Example $a_n = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.

If $F(x) = \log(x)$ then $\frac{dF}{dx} = \frac{1}{x}$ and

$$\int_a^b \frac{1}{x}\,dx = \log(b) - \log(a).$$

$= (\text{area under } y = \frac{1}{x} \text{ from } x = a \text{ to } x = b)$



So $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

= area of the shaded boxes

> area under $y = \frac{1}{x}$ from $x = 1$ to $x = 10000000000$

$= \log(10000000000000) - \log(1)$

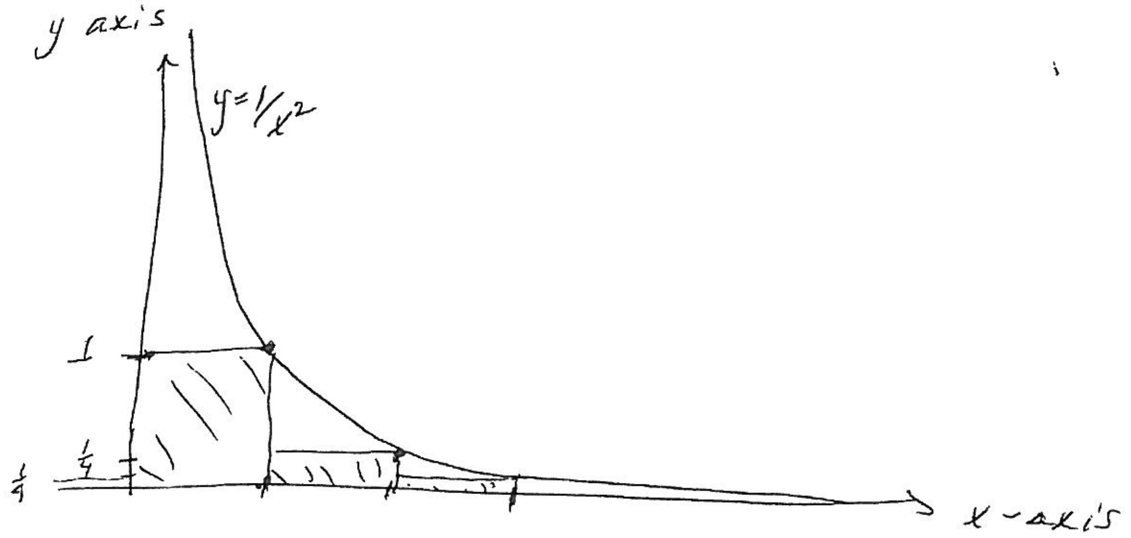
= VERY LARGE.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example $(a_n) = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

If $F(x) = \frac{-1}{x}$ then $\frac{dF}{dx} = \frac{1}{x^2}$. and

$\int_a^b \frac{1}{x^2}\,dx = \frac{-1}{b} - \left(\frac{-1}{a}\right) =$ area under $y = \frac{1}{x^2}$ between $x = a$ and $x = b$



Then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

= area of shaded boxes

$<$ 1 + (area under $y = \frac{1}{x^2}$ from $x = 1$ to $x = 1000000000$)

$$= 1 + \left(\frac{-1}{1000000000} - \frac{-1}{1}\right)$$

$= 1 + 1 - \frac{1}{\cdots}$ is very close to 2.

So $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.