

MATH 221 Lecture 11, October 2, 2000 (1)

$\lim_{x \rightarrow 2} f(x) = 10$ if $f(x)$ gets closer and closer to 10 as x gets closer and closer to 2.

Example Evaluate $\lim_{x \rightarrow 2} \frac{3x^2+8}{x^2-x}$.

When $x=1$, $\frac{3x^2+8}{x^2-x} = 11$.

When $x=1.5$, $\frac{3x^2+8}{x^2-x} = 19.66\dots$

When $x=1.9$, $\frac{3x^2+8}{x^2-x} = 11.011\dots$

When $x=1.99$, $\frac{3x^2+8}{x^2-x} = 10.091\dots$

When $x=1.999$, $\frac{3x^2+8}{x^2-x} = 10.00901\dots$

When $x=1.9999$, $\frac{3x^2+8}{x^2-x} = 10.0009001\dots$

Comment -
Experimenting by trying out values often shows what the limit will be. Here denominator and numerator are continuous, so we can put in the limit value 2.

So $\lim_{x \rightarrow 2} \frac{3x^2+8}{x^2-x} = 10$

Usually determining the limit is straightforward

Example $\lim_{x \rightarrow 1} 6x^2 - 4x + 3 = 5$

But sometimes...

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Example $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} \stackrel{?}{=} \frac{0}{0}$

$\frac{0}{0}$ makes NO SENSE.

Example $\lim_{x \rightarrow 0} \frac{5x}{x} \stackrel{?}{=} \frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{5x}{x} = \lim_{x \rightarrow 0} 5 = 5.$$

Example $\lim_{x \rightarrow 0} \frac{17x}{2x} \stackrel{?}{=} \frac{0}{0}.$

$$\lim_{x \rightarrow 0} \frac{17x}{2x} = \lim_{x \rightarrow 0} \frac{17}{2} = \frac{17}{2}.$$

Let's go back to

Example Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x}$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x}+1} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Comment We can try to simplify a fraction before we take the limit. So often the quantity going to zero can be "cancelled out"

So, whenever a limit looks like it is coming ^③ out to $\frac{0}{0}$ it needs to be looked at in a different way to see what it is really getting closer and closer to.

Example $\lim_{x \rightarrow 7} \frac{x^2 - 49}{x - 7} = \lim_{x \rightarrow 7} \frac{(x-7)(x+7)}{(x-7)}$

$$= \lim_{x \rightarrow 7} (x+7) = 7+7 = 14.$$

Example Evaluate $\lim_{x \rightarrow 5} \frac{x^5 - 3125}{x - 5}$.

$$\lim_{x \rightarrow 5} \frac{x^5 - 3125}{x - 5} = \lim_{x \rightarrow 5} \frac{x^5 - 5^5}{x - 5}$$

$$= \lim_{x \rightarrow 5} \frac{(x-5)(x^4 + 5x^3 + 5^2x^2 + 5^3x + 5^4)}{x - 5}$$

$$= \lim_{x \rightarrow 5} x^4 + 5x^3 + 5^2x^2 + 5^3x + 5^4$$

$$= 5^4 + 5^4 + 5^4 + 5^4 + 5^4 = 5^5 = 3125.$$

Example Evaluate $\lim_{x \rightarrow a} \frac{x^{5/2} - a^{5/2}}{x - a}$.

Comment

More examples where quantity going to zero cancels out.
L'Hôpital's rule can assist if stuck. Best to try other methods first.

$$\lim_{x \rightarrow a} \frac{x^{5/2} - a^{5/2}}{x - a} = \lim_{x \rightarrow a} \frac{(x^{5/2} - a^{5/2})}{(x - a)} \frac{(x^{5/2} + a^{5/2})}{(x^{5/2} + a^{5/2})} \quad (4)$$

$$= \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} \cdot \frac{1}{x^{5/2} + a^{5/2}}$$

$$= \lim_{x \rightarrow a} \frac{(x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{x - a} \cdot \frac{1}{x^{5/2} + a^{5/2}}$$

$$= \lim_{x \rightarrow a} \frac{x^4 + ax^3 + a^2x^2 + a^3x + a^4}{x^{5/2} + a^{5/2}}$$

$$= \frac{a^4 + a^4 + a^4 + a^4 + a^4}{a^{5/2} + a^{5/2}} = \frac{5a^4}{2a^{5/2}} = \frac{5}{2} a^{3/2}$$

Particularly useful limits

$$(a) \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots}{x}$$

$$= \lim_{x \rightarrow 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \dots$$

$$= 1 - 0 + 0 - 0 + 0 - \dots = 1$$

$$(b) \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots) - 1}{x}$$

Note: We need to know this is a nice convergent series to substitute in $x = 0$.

$$= \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots}{x} \quad (5)$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \dots}{x}$$

$$= 0 + 0 + 0 + \dots = 0.$$

$$(c) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots) - 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x}$$

$$= \lim_{x \rightarrow 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots = 1 + 0 + 0 + \dots$$

$$= 1.$$

Again need
good convergence of
our series

$$(d) \text{ Evaluate } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}.$$

Let $y = \ln(1+x)$. Then $e^y = 1+x$ and $x = e^y - 1$.

Also $y \rightarrow 0$ as $x \rightarrow 0$. So

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{y \rightarrow 0} \frac{y}{e^y - 1} = \lim_{y \rightarrow 0} \frac{1}{\frac{e^y - 1}{y}}$$

$$= \frac{1}{1} = 1.$$

Comment We are really
using f continuous, g
continuous $\Rightarrow f \circ g$ is
continuous here

Example Evaluate $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$.

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$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{x \rightarrow 0} (e^{\ln(1+x)})^{\frac{1}{x}} = \lim_{x \rightarrow 0} e^{\frac{1}{x} \ln(1+x)}$$

$$= \lim_{x \rightarrow 0} e^{\frac{\ln(1+x)}{x}} = e^1 = e.$$

Using e^x is a continuous function

Example Evaluate $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.

Let $x = \frac{1}{n}$. Then $x \rightarrow 0$ as $n \rightarrow \infty$.

$$\text{So } \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$$

Note: $n \rightarrow \infty$ means n gets larger and larger.

Example Evaluate $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$

Let $y = x - \pi$. Then $y \rightarrow 0$ as $x \rightarrow \pi$.

$$\text{So } \lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} = \lim_{y \rightarrow 0} \frac{\sin(y + \pi)}{y} = \lim_{y \rightarrow 0} \frac{\sin y \cos \pi + \cos y \sin \pi}{y}$$

$$= \lim_{y \rightarrow 0} \frac{0 + (-1) \cos y}{y} = \lim_{y \rightarrow 0} \frac{\sin y (-1) + \cos y \cdot 0}{y}$$

$$= \lim_{y \rightarrow 0} \frac{-\sin y}{y} = -1.$$