Lecture 11 24 March 2010 ①

The interest sequence

Example If you borrow \$500 on your credit card at 14% interest, find the amounts due at the end of two years if the interest is compounded

(a) annually, (e) hourly,

(b) quarterly, (f) every second,

(c) monthly, (g) every nanosecond,

(d) daily, (h) continuously.

(a) You owe

$$500 + 500(.14) = 500(1+.14) \text{ after one year}$$

and $500(1+.14)(1+.14)$ after two years.

(c) You owe

$$500 + 500\left(\frac{.14}{12}\right) = 500\left(1+\frac{.14}{12}\right) \text{ after one month}$$

$$500\left(1+\frac{.14}{12}\right)\left(1+\frac{.14}{12}\right) \text{ after two months}$$

$$500\left(1+\frac{.14}{12}\right)^{24} \text{ after two years.}$$

(f) You owe

$$500 + 500\left(\frac{.14}{365\cdot 24\cdot 3600}\right) \text{ after 1 second}$$

and $$500\left(1+\frac{.14}{365\cdot 24\cdot 3600}\right)^{2\cdot 365\cdot 24\cdot 3600}$$

after two years.

1h) You owe

$$\lim_{n\to\infty} 500\left(1+\frac{.14}{n}\right)^{2n} \text{ after 2 years.}$$

$$\lim_{n\to\infty} 500\left(1+\frac{.14}{n}\right)^{2n} = 500 \lim_{n\to\infty} e^{\log\left(1+\frac{.14}{n}\right)^{2n}}$$

$$= 500 \lim_{n\to\infty} e^{2n\log\left(1+\frac{.14}{n}\right)} = 500 \lim_{n\to\infty} e^{2\cdot(.14)\frac{\log\left(1+\frac{.14}{n}\right)}{.14/n}}$$

$$= 500 \lim_{n\to\infty} e^{.28\frac{\log\left(1+\frac{.14}{n}\right)}{.14/n}}$$

Recall

$$\cancel{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - x^4}$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

So you owe

$$500\cdot e^{.28} \text{ after two years.}$$

<u>Note:</u>

$$500(1+.14)^2 = 649.80$$

$$500\left(1+\frac{.14}{12}\right)^{24} \approx 660.49$$

$$500e^{.28} \approx 661.56$$

Newton Iteration Solving $f(x)=0$

The Taylor series of $f(x)$ at $x=a$ is

$$f(x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \ldots$$

Then, for nice functions f (functions that don't jump around much)

$$f(x) \simeq f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) = f(a) + f'(a)(x-a).$$

Create a sequence:

$$z_0 = \text{your choice}$$

$$z_1 = \frac{-f(z_0)}{f'(z_0)} + z_0,$$

$$z_2 = \frac{-f(z_1)}{f'(z_1)} + z_1,$$

$$\vdots$$

Then, if $\lim_{n\to\infty} z_n = z$ (the sequence z_n converges)

then $f(z_{n+1})$ is very close to

$$f(z_n) + f'(z_n)(z_{n+1} - z_n) = f(z_n) + f'(z_n)\left(\frac{-f(z_n)}{f'(z_n)} + z_n - z_n\right) = 0$$

So $f(z)=0$.

This "trick" for solving $f(z)=0$ totally fails if $f'(z_n)$ ever comes out 0 (or very very small) or if f jumps around wildly.

Picard iteration Solving $f(x) = x$.

Create a sequence (a_n):

$a_1 =$ your choice

$a_2 = f(a_1)$,

$a_3 = f(a_2)$,

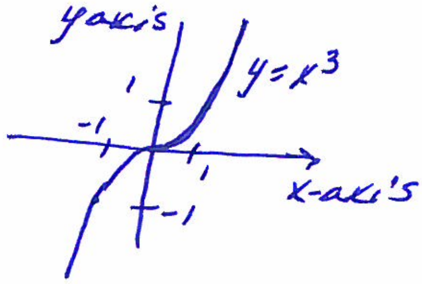
$\vdots$

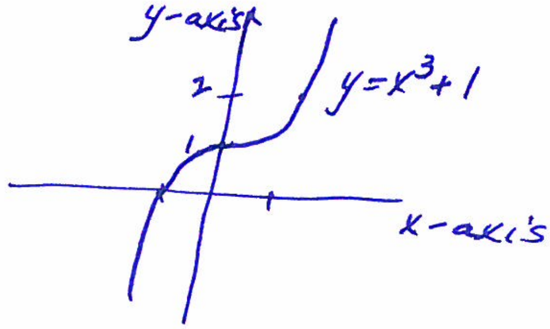
Then, if $\lim_{n\to\infty} a_n = a$ (the sequence (a_n) converges)

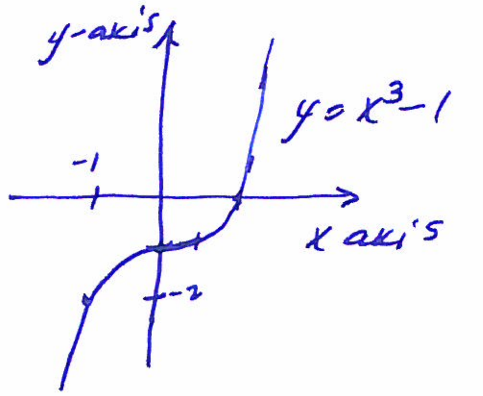
then $f(a) = a$ (because $f(a_n) = a_{n+1}$ is very close to a_n for large enough n)

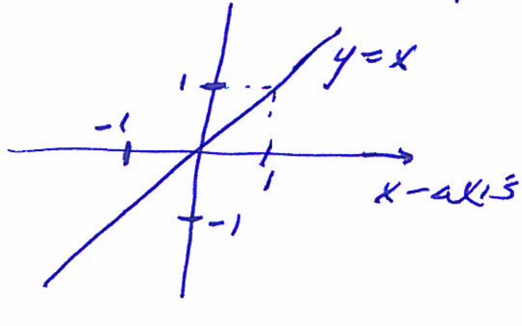
Example Show that the equation $x^3 + x - 1 = 0$ has a solution between 0 and 1.

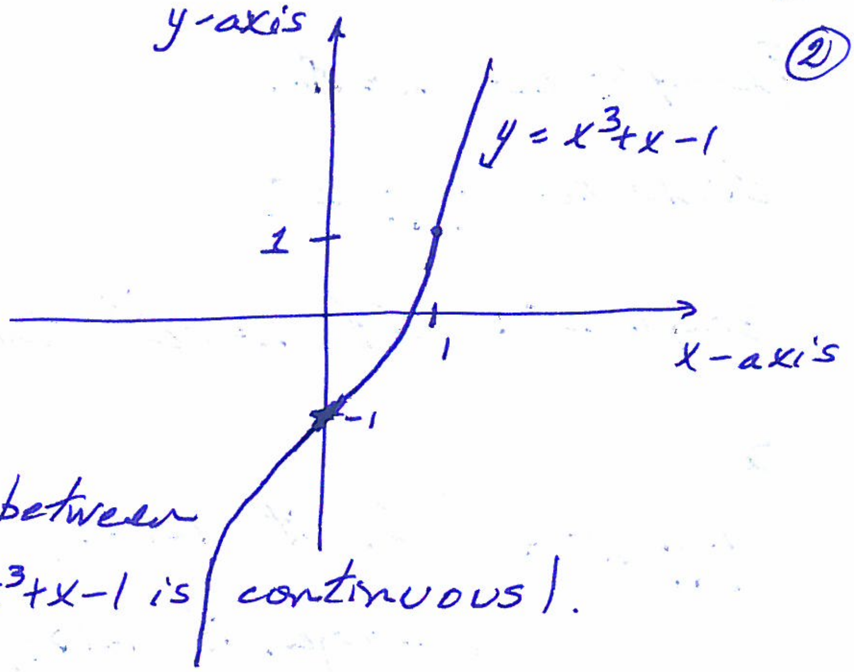
Graphs:











Notes

(a) If $x=0$ then
$x^3+x-1=-1$

(b) If $x=1$ then
$x^3+x-1=1+1-1=1$

So x^3+x-1 has a zero between $x=0$ and $x=1$ (because x^3+x-1 is continuous).

Example

Transform the equation $x^3+x-1=0$ to the form $x=f(x)$ and use Picard iteration to find the solution to 3 decimal places.

Since $x^3+x-1=0$ is the same as $x(x^2+1)=1$

$$x=\frac{1}{x^2+1} \text{ is of the form } x=f(x).$$

Let $a_1=\frac{1}{2}$. Then

$$a_2=\frac{1}{\left(\frac{1}{2}\right)^2+1}=\frac{1}{\frac{1}{4}+1}=\frac{1}{\frac{5}{4}}=\frac{4}{5}=0.8$$

$$a_3=\frac{1}{\left(\frac{4}{5}\right)^2+1}=\frac{1}{\frac{16}{25}+1}=\frac{1}{\frac{41}{25}}=\frac{25}{41}\approx.\overline{60975}\,\overline{60975}$$

$$a_4=\frac{1}{\left(\frac{25}{41}\right)^2+1}=\frac{1}{\frac{625}{1681}+1}=\frac{1}{\frac{625+1681}{1681}}=\frac{1681}{2306}\approx.728967$$

$a_5\approx.6530046$

$a_6\approx.7010582$

$a_7\approx.6704737$

$a_8\approx.68987635$

$a_9\approx.67753918$

$a_{10}\approx.68537308$

$a_{11}\approx.680394233$

$a_{12}\approx.6835567$

$a_{13} \approx .681547222$ $\quad a_{14} \approx .68282382$

$a_{15} \approx .6820126$

So, to 3-decimal places of accuracy

$x = .682$ is a zero of $x^3 + x - 1 = 0$.

<u>Note</u> Another expression for $x^3 + x - 1 = 0$ is

$x = 1 - x^3$.

Let $a_1 = \frac{1}{2}$

$a_2 = 1 - \left(\frac{1}{2}\right)^3 = 1 - \frac{1}{8} = \frac{7}{8} = .875$

$a_3 = 1 - \left(\frac{7}{8}\right)^3 \cancel{\approx .001953} \approx .330078$

$a_4 \approx \cancel{1-(.001953)^3 \approx} \; 1 - (.330078)^3 \approx .964037$

$a_5 \approx .104055,$

$\vdots$

and this sequences is not converging,

but oscillating between ~~from~~ close to 1 and ~~to~~ close to 0.