Lect 10, ~~[illegible]~~ 22 March 2010 0

Sequences

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1. Sequences

Let Y be a set. A **sequence** $(y_1, y_2, y_3, \ldots)$ in Y is a function

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \longrightarrow & Y \\ n & \longmapsto & y_n. \end{array}$$

Let Y $\subseteq \mathbb{R}$ ~~be a set with a partial order $\le$~~. Let $(y_1, y_2, \ldots)$ be a sequence in Y. The sequence $(y_1, y_2, \ldots)$ is **increasing** if $(y_1, y_2, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then } y_i \le y_{i+1}.$$

The sequence $(y_1, y_2, \ldots)$ is **decreasing** if $(y_1, y_2, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then } y_i \ge y_{i+1}.$$

A sequence (a_n) is monotone if (a_n) is increasing or decreasing.

Let Y be a metric space. Let $(y_1, y_2, \ldots)$ be a sequence in Y. The sequence is **bounded** if the set

$$\{y_1, y_2, \ldots\} \text{ is bounded.}$$

The sequence $(y_1, y_2, \ldots)$ is **contractive** if $(y_1, y_2, \ldots)$ satisfies: There exists $\alpha \in (0, 1)$ such that

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then } d(y_i, y_{i+1}) \le \alpha d(y_{i-1}, y_i).$$

The sequence $(y_1, y_2, \ldots)$ is **Cauchy** if $(y_1, y_2, \ldots)$ satisfies

> if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(y_m, y_n) < \varepsilon$.

Let $l \in Y$. The sequence $(y_1, y_2, \ldots)$ **converges** to l if $(y_1, y_2, \ldots)$ satisfies

> if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(y_n, l) \le \varepsilon$.

Let $(y_1, y_2, \ldots)$ be a sequence in $\mathbb{R}$ (or more generally, any totally ordered set with the order topology). The **upper limit** of $(y_1, y_2, \ldots)$ is

$$\limsup y_n = \lim_{n\to\infty} \sup\{y_n, y_{n+1}, \ldots\}.$$

The **lower limit** of $(y_1, y_2, \ldots)$ is

$$\liminf y_n = \lim_{n\to\infty} \inf\{y_n, y_{n+1}, \ldots\}.$$

Example: If $y_n = (-1)^n\left(1 - \frac{1}{n}\right)$ then

$$\limsup y_n = 1 \qquad \text{and} \qquad \liminf y_n = -1.$$

Proposition 1.1 *Let $(y_1, y_2, \ldots)$ be a sequence in $\mathbb{R}$. Then*

a. $\limsup y_n = \sup\{\text{cluster points of } (y_1, y_2, \ldots)\}$, *and*
b. $\liminf y_n = \inf\{\text{cluster points of } (y_1, y_2, \ldots)\}$.

2. References [PLACEHOLDER]

[BG] A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmanian*, Duke Math. J. **107** no. 3, (2001), 561-575; arXiv:math/9909077v2, MR1828302 (2002e:20083)

Boundedness, sup, inf, limsup and liminf. ①

Example Let $a_n = \frac{\log n}{\sqrt{n}}$.

(a) Show that a_n is bounded

(b) Find $N \in \mathbb{Z}_{>0}$ such that a_n is decreasing for $n \in \mathbb{Z}_{>0}$ such that $n > N$.

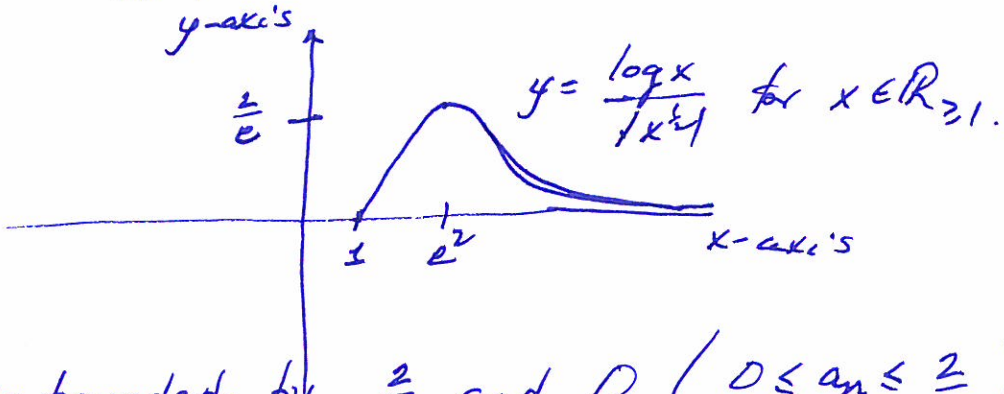
(c) Show that $\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0$.

Since
$$\frac{d}{dx}\left(\frac{\log x}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-\frac{1}{2}}\log x) = x^{-\frac{1}{2}}\frac{1}{x} + \frac{-1}{2}x^{-3/2}\log x$$
$$= x^{-3/2}(1-\tfrac{1}{2}\log x) = \tfrac{1}{2}x^{-3/2}(2-\log x)$$
is less than 0 for $x > e^2$,
is equal to 0 for $x = e^2$, and
is greater than 0 for $x < e^2$,

the function $f(x) = \frac{\log(x)}{|x^{\frac{1}{2}}|}$ is decreasing for $x > e^2$ and has a maximum at e^2.

Since $\log(x) \geq 0$ for $x \geq 1$ and $|x^{\frac{1}{2}}| > 0$ for $x \geq 1$,

$$a_n = \frac{\log n}{|n^{\frac{1}{2}}|} \geq 0 \text{ for } n \in \mathbb{Z}_{>0}$$



So a_n is bounded by $\frac{2}{e}$ and 0 $\left(0 \leq a_n \leq \frac{2}{e}\right)$ and a_n is decreasing if $n > e^2$ (in particular, if $n > 9$).

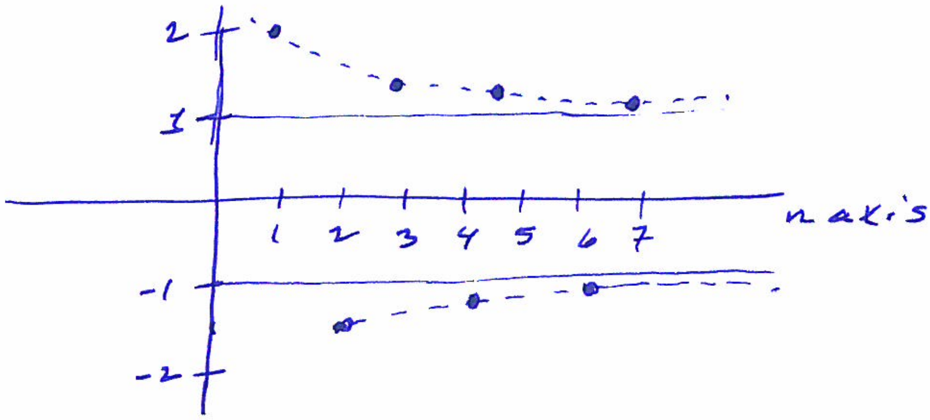
(c) $$\lim_{n\to\infty} \frac{\log n}{n^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{\log n}{(e^{\log n})^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{\log n}{e^{\frac{1}{2}\log n}}$$

$$= \lim_{y\to\infty} \frac{y}{e^{\frac{1}{2}y}} = \lim_{y\to\infty} \frac{y}{1+\frac{1}{2}y+\frac{1}{2!}(\frac{1}{2}y)^2+\cdots}$$

$$= \lim_{y\to\infty} \frac{1}{\frac{1}{y}+\frac{1}{2}+\frac{1}{2!}\frac{1}{2^2}y+\frac{1}{3!}\frac{1}{2^3}y^2+\cdots}$$

$$\leq \lim_{y\to\infty} \frac{1}{\frac{1}{2!}\frac{1}{2^2}y} = \lim_{y\to\infty} \frac{8}{y} = 0.$$

<u>Example</u> Analyse the sequence $a_n = (-1)^{n-1}(1+\frac{1}{n})$.



a_n is bounded above by 2

a_n is bounded below by -2

$\limsup a_n = 1$ and $\liminf a_n = -1$

$\sup a_n = 2$ and $\inf a_n = -\frac{3}{2}$.

$\lim_{n\to\infty} a_n$ does not exist because, as n gets larger and larger, a_n oscillates between close to $\limsup a_n$ (1, in this case) and $\liminf a_n$ (-1, in this case)

<u>Theorem</u> If a_n is a sequence in $\mathbb{R}$ (or an totally ordered set X) such that

(a) a_n is increasing,

(b) a_n is bounded, and

(c) $\sup a_n$ exists,

then (a_n) converges to $\sup a_n$.

<u>Note</u>: In $\mathbb{R}$, ~~s~~ $\sup a_n$ always exists,

in $\mathbb{Q}$, $\sup a_n$ does not always exist.

<u>Proof</u> Let $l = \sup a_n$. To show: $\lim_{n \to \infty} a_n = l$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, l) < \varepsilon$.

Proof by contradiction.

Assume that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that there does not exist $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, l) < \varepsilon$.

So, if $N \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ with $n > N$ such that $d(a_n, l) > \varepsilon$.

So, if $N \in \mathbb{Z}_{>0}$ then $l - \varepsilon > a_n \geq a_N$.

So $l - \varepsilon$ is an upper bound of (a_n).

Contradiction to $l = \sup a_n$

So $\lim_{n \to \infty} a_n = l$. //.