Limit examples for 19 March

Example Prove that $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

Proof

$$\lim_{x\to 0} \left|\frac{e^x-1}{x} - 1\right| = \lim_{x\to 0}\left|\frac{e^x-1-x}{x}\right|$$

$$= \lim_{x\to 0}\left|\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots -1-x}{x}\right|$$

$$= \lim_{x\to 0}\left|\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots\right| = \lim_{x\to 0} |x|\left|\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right|$$

$$\le \lim_{x\to 0} |x|\left(\frac{1}{2!}+\frac{|x|}{3!}+\frac{|x|^2}{4!}+\cdots\right)$$

$$\le \lim_{x\to 0} |x|\left(1+|x|+|x|^2+\cdots\right)$$

$$= \lim_{x\to 0} |x|\frac{1}{1-|x|} = 0\cdot 1 = 0.$$

So $\lim_{x\to 0}\frac{e^x-1}{x} = 1$.

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Example Evaluate $\lim_{x\to 0}\frac{\cos x - 1}{x^2}$.

$$\lim_{x\to 0}\frac{\cos x-1}{x^2} = \lim_{x\to 0}\frac{\left(\frac{e^{ix}+e^{-ix}}{2}-1\right)}{x^2} = \lim_{x\to 0}\frac{1}{2}\left(\frac{e^{ix}+e^{-ix}-2}{x^2}\right)$$

$$= \lim_{x\to 0}\frac{1}{2}e^{-ix}\frac{\left(e^{2ix}-2e^{ix}+1\right)}{x^2} = \lim_{x\to 0}\frac{-1}{2}e^{-ix}\frac{\left(e^{ix}-1\right)^2}{(ix)^2} = \frac{-1}{2}\cdot e^0\cdot 1^2$$

$$= \frac{-1}{2}.$$

Example Find $\lim_{x\to 2} \frac{3x^2+8}{x^2-x}$

$$\lim_{x\to 2} \frac{3x^2+8}{x^2-x} = \left(\lim_{x\to 2} 3x^2+8\right)\left(\lim_{x\to 2} \frac{1}{x^2-x}\right)$$

$$= (3\cdot 2^2+8)\frac{1}{2^2-2} = (3\cdot 4+8)\frac{1}{2} = \frac{20}{2} = 10,$$

since $\lim_{x\to 2} 3x^2+8$ and $\lim_{x\to 2} \frac{1}{x^2-x}$ exist

(because $f(x)=3x^2+8$ and $f(x)=\frac{1}{x^2-x}$ are continuous at $x=2$ — the graph doesn't jump at $x=2$.)

Example Find $\lim_{x\to 0} \frac{5x}{x}$

$\lim_{x\to 0} \frac{5x}{x} = \left(\lim_{x\to 0} 5x\right)\left(\lim_{x\to 0} \frac{1}{x}\right)$ NO. We can only do this if $\lim_{x\to 0} 5x$ exists and $\lim_{x\to 0} \frac{1}{x}$ exists.

A more sensible approach is to use some algebra,

$$\lim_{x\to 0} \frac{5x}{x} = \lim_{x\to 0} \frac{5}{1} = 5.$$

<u>Example</u> Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x}-1)}{x}\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$$

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<u>Example</u> Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{e^{ix}-e^{-ix}}{2ix} = \lim_{x\to 0} \frac{\frac{e^{ix}-1}{ix} - \frac{e^{-ix}-1}{ix}}{2}$$

$$= \lim_{x\to 0} \frac{1}{2}\left(\frac{e^{ix}-1}{ix} + \frac{e^{-ix}-1}{-ix}\right) = \frac{1}{2}(1+1) = 1.$$

<u>Example</u> $\lim_{x\to 0} \frac{\log(1+x)}{x}$

Let $x = e^y - 1$. Then $y = \log(x+1)$ and $y \to 0$ as $x \to 0$.

So

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{y\to 0} \frac{\log(1+e^y-1)}{e^y-1} = \lim_{y\to 0} \frac{\log(e^y)}{e^y-1}$$

$$= \lim_{y\to 0} \frac{y}{e^y-1} = \lim_{y\to 0} \frac{1}{\frac{e^y-1}{y}} = \frac{1}{1} = 1.$$

__Example__ Evaluate $\lim_{n\to\infty}\left(1+\frac{34}{n}\right)^n$.

$$\lim_{n\to\infty}\left(1+\frac{34}{n}\right)^n = \lim_{n\to\infty} e^{\log\left(1+\frac{34}{n}\right)^n} = \lim_{n\to\infty} e^{n\log\left(1+\frac{34}{n}\right)}$$

$$= \lim_{n\to\infty} e^{34\cdot\frac{\log\left(1+\frac{34}{n}\right)}{34/n}} = e^{34\cdot 1} = e^{34}.$$

Theorem Let a_n and b_n be sequences in $\mathbb{R}$.
Assume that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ exists.

If $a_n \le b_n$ then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

Proof Proof by contradiction.

Let $l_1 = \lim_{n\to\infty} a_n$ and $l_2 = \lim_{n\to\infty} b_n$.

Assume $l_1 > l_2$. Let $\varepsilon = l_1 - l_2$

Let $N_1 \in \mathbb{Z}_{>0}$ be such that

if $n \in \mathbb{Z}_{>0}$ and $n > N_1$ then $|a_n - l_1| < \varepsilon/2$.

Let $N_2 \in \mathbb{Z}_{>0}$ be such that

if $n \in \mathbb{Z}_{>0}$ and $n > N_2$ then $|b_n - l_1| < \varepsilon/2$.

Let $N \in \mathbb{Z}_{>0}$ be such that $N > N_1$ and $N > N_2$.

Then

$$a_N > l_1 - \varepsilon/2 = l_2 + \varepsilon - \varepsilon_2 = l_2 + \varepsilon/2 > b_N.$$

This is a contradiction to $a_N \le b_N$.

So $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

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__Theorem__ Assume that $\lim_{x\to a} g(x) = \ell$

and $\lim_{y\to \ell} f(y)$ exists. Then

$$\lim_{y\to\ell} f(y) = \lim_{x\to a} f(g(x))$$

__Proof__ Let $L = \lim_{y\to\ell} f(y)$.

To show: $\lim_{x\to a} f(g(x)) = L$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta\in\mathbb{R}_{>0}$ such that

if $|x-a|<\delta$ then $|f(g(x))-L|<\varepsilon$.

Assume $\varepsilon\in\mathbb{R}_{>0}$.

To show: There exists $\delta\in\mathbb{R}_{>0}$ such that

if $|x-a|<\delta$ then $|f(g(x))-L|<\varepsilon$.

Let $\delta_1\in\mathbb{R}_{>0}$ be such that

if $|y-\ell|<\delta_1$ then $|f(y)-L|<\varepsilon$.

Let $\delta\in\mathbb{R}_{>0}$ be such that

if $|x-a|<\delta$ then $|g(x)-\ell|<\delta_1$

~~Then~~ So

if $|x-a|<\delta$ then

$|g(x)-\ell|<\delta_1$ and $|f(g(x))-L|<\varepsilon$. //