Notes for 17 March

Recall

Theorem Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

(a) $\lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

(c) If $c\in\mathbb{R}$ then $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

(d) If $f(x)\le g(x)$ then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(e) If $g(x)\neq 0$ then $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{\lim_{x\to a} g(x)}$

(f) If $\lim_{x\to a} g(x) = l$ then $\lim_{y\to l} f(y) = \lim_{x\to a} f(g(x))$.

For lecture 8, 17 March presentation

Theorem (a) Let $x\in\mathbb{C}$. Then

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|>1 \\ 1, & \text{if } x=1 \\ \text{diverges}, & \text{if } |x|=1 \text{ and } x\neq 1. \end{cases}$$

(b) Let $x\in\mathbb{C}$. Then

$$\lim_{n\to\infty} 1+x+\cdots+x^n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x}, & \text{if } |x|<1, \\ \text{diverges}, & \text{if } |x|\ge 1. \end{cases}$$

Theorem (a) Let $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{C}$. Then

$$\lim_{x \to a} x^n = a^n.$$

(b) Let $a \in \mathbb{C}$. Then

$$\lim_{x \to a} e^x = e^a.$$

<u>Our definitions</u>

$\lim_{x \to a} f(x) = \ell$ if $f(x)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $d(x,a) < \delta$ then $d(f(x), \ell) < \varepsilon$.

In $\mathbb{R}^n$, $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is given by

$$d(x,y) = \left| \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \right|$$

where

$$|a| = \sup(a, -a), \text{ for } a \in \mathbb{R}$$

We defined $\sqrt{x}$ as the inverse expression to x^2 so that branches are possible and $\sqrt{9} = -3$ is possible.

We defined $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

The limits $\lim_{n\to\infty} x^n$ and $\lim_{n\to\infty} 1+x+x^2+\cdots+x^n$

Theorems

(a) Let $x \in \mathbb{C}$.

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|>1 \\ 1, & \text{if } x=1 \\ \text{diverges}, & \text{if } |x|=1 \text{ and } x\neq 1 \end{cases}$$

(b) Let $x \in \mathbb{C}$.

$$\lim_{n\to\infty} 1+x+x^2+\cdots+x^n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

$$= \begin{cases} \frac{1}{1-x}, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|\geq 1 \end{cases}$$

Proof

If $x=1$ then the sequence $a_n = x^n$ is $a_n = 1$ and $\lim_{n\to\infty} 1^n = 1$.

If $x=1$ then the $\lim_{n\to\infty} 1+1^2+\cdots+1^n = \lim_{n\to\infty} n$, which diverges.

The remaining statements in (b) follow from (a).

<u>Example</u> Let $x \in \mathbb{C}$ with $|x|<1$.

Prove that $\lim_{n\to\infty} x^n = 0$.

<u>Proof</u> Let $N \in \mathbb{Z}_{>0}$ such that $|x| < 1 - \frac{1}{N+1}$.

$$\lim_{n\to\infty} |x^n - 0| = \lim_{n\to\infty} |x|^n \le \lim_{n\to\infty} \left(1 - \frac{1}{N+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{N+1-1}{N+1}\right)^n = \lim_{n\to\infty} \left(\frac{N}{N+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{N}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{1 + n\frac{1}{N} + \cdots + \left(\frac{1}{N}\right)^n} \le \lim_{n\to\infty} \frac{1}{1+\frac{n}{N}}$$

$$= \lim_{n\to\infty} \frac{N}{n+N} = N \lim_{n\to\infty} \frac{1}{n+N} = N \cdot 0 = 0.$$

<u>Example</u> Let $x \in \mathbb{C}$ with $|x| > 1$.

Prove that $\lim_{n\to\infty} x^n$ diverges.

<u>Proof</u> Let $N \in \mathbb{Z}_{>0}$ be such that $|x| > 1 + \frac{1}{N}$.

Then
$$|x|^n > \left(1+\frac{1}{N}\right)^n = 1 + n\left(\frac{1}{N}\right) + \cdots + \left(\frac{1}{N}\right)^n$$

$$> n\left(\frac{1}{N}\right) = \frac{n}{N}$$

Since $\frac{n}{N}$ is unbounded as n gets larger and larger,

$|x|^n$ is unbounded as $n \to \infty$.

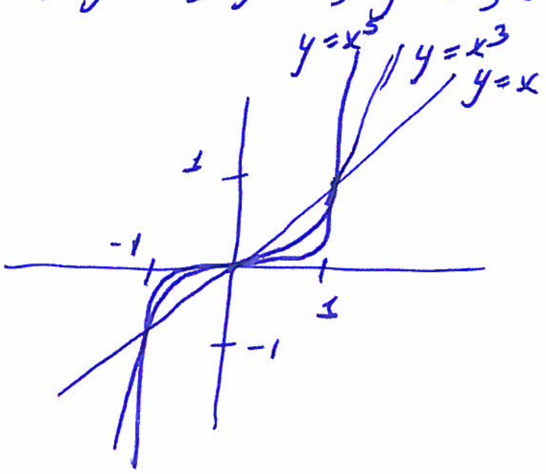
So $\lim_{n\to\infty} x^n$ diverges.

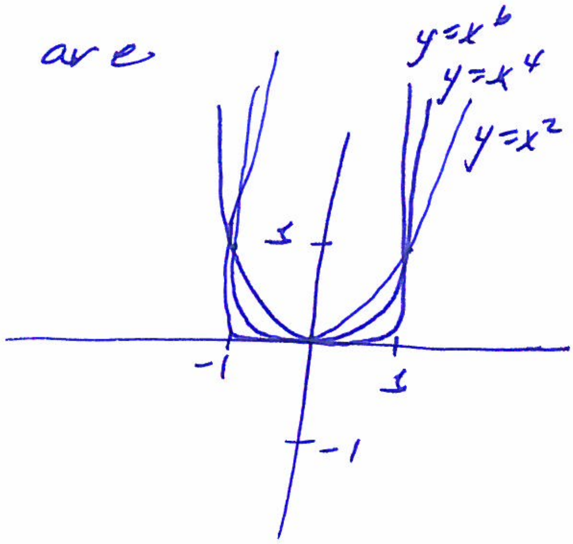
<u>Example</u> Let $x \in \mathbb{R}$ with $|x|<1$. Let $a_n = x^n$.

Find $\lim_{n\to\infty} a_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x^n$ and the graphs of

$y=x$, $y=x^2$, $y=x^3$, $y=x^4$, ... are





So $\lim_{n\to\infty} x^n = 0$ when $|x|<1$.

$\lim_{n\to\infty} x^n$ diverges when $|x|>1$

$\lim_{n\to\infty} 1^n = 1$ and $\lim_{n\to\infty} (-1)^n$ diverges.

<u>Example</u> Let $x \in \mathbb{R}$. Find $\lim_{n\to\infty} 1+x+x^2+\cdots+x^n$.

$$\lim_{n\to\infty} 1+x+x^2+\cdots+x^n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} \quad \text{if } |x|<1$$

For example, if $x=\frac{1}{2}$

$$\lim_{n\to\infty} 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^n = \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}} = 2.$$

Let $n \in \mathbb{Z}_{>0}$.

__Example__ Let $a \in \mathbb{R}$. Prove that $\lim_{x \to a} x^n = a^n$.

__Proof__: To show: $\lim_{y \to 0} |(y+a)^n - a^n| = 0$.

$$\lim_{y\to 0} |(y+a)^n - a^n| = \lim_{y \to 0} | y^n + ny^{n-1}a + \cdots + na^{n-1}y + a^n - a^n|$$

$$= \lim_{y \to 0} |y^n + ny^{n-1}a + \cdots + na^{n-1}y|$$

$$\leq \lim_{y \to 0} |y(y^{n-1} + ay^{n-2}n + \cdots + na^{n-1})|$$

$$= \lim_{y \to 0} |y|\,|y^{n-1} + any^{n-2} + \cdots + na^{n-1}|$$

$$\leq \lim_{y\to 0} |y|(|y|^{n-1} + |a|n|y|^{n-2} + \cdots + |na^{n-1}|)$$

$$\leq \lim_{y \to 0} |y| n |a|^{n-1} = 0 \cdot n \cdot |a|^{n-1} = 0. \quad //$$

So $\underline{f(x) = x^n \text{ is continuous at } x = a}$.

An alternative proof is that

(a) $f(x) = x$ (the identity function) is continuous,

(b) the product is continuous (since $\mathbb{R}$ is a topological field)

and therefore $\lim_{x \to a} x^n = a^n$.

<u>Example</u> Prove that $\lim_{x\to 0} e^x = e^0$.

<u>Proof.</u>

$$\lim_{x\to 0}\left|\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1\right|$$

$$=\lim_{x\to 0}\left|x\left(1+\frac{x}{2}+\frac{x^2}{3!}+\cdots\right)\right|$$

$$\leq \lim_{x\to 0} |x|\left(1+\frac{|x|}{2}+\frac{|x|^2}{3!}+\cdots\right)$$

$$\leq \lim_{x\to 0} |x|\left(1+|x|+|x|^2+\cdots\right)$$

$$=\lim_{x\to 0} |x|\cdot\frac{1}{1-|x|} = 0\cdot 1 = 0.$$

<u>Example</u> Prove that $\lim_{x\to a} e^x = e^a$.

<u>Proof.</u>

$$\lim_{x\to a} e^x = \lim_{y\to 0} e^{y+a} = \lim_{y\to 0} e^a e^y \cancel{= \lim}$$

$$= e^a \lim_{y\to 0} e^y = e^a\cdot e^0 = e^{a+0} = e^a. //$$

<u>Theorem</u>

Hence <u>e^x is continuous at $x=a$.</u>