Problem Set: Limits
620-295 Semester I 2010

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1. Evaluating Limits when $x \to 0$

(1) Evaluate $\lim_{x \to 0} \left(x^2 - 2\right)^2 + 6$.

(2) Evaluate $\lim_{x \to 0} \frac{5x}{x}$.

(3) Evaluate $\lim_{x \to 0} \frac{17x}{2x}$.

(4) Evaluate $\lim_{x \to 0} \frac{-317x}{422x}$.

(5) Evaluate $\lim_{x \to 0} \frac{-317x - 3}{422x + 5}$.

(6) Evaluate $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

(7) Evaluate $\lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x}$.

(8) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{2+x} - \sqrt{2}}{x}$.

(9) Evaluate $\lim_{h \to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}\right)$.

(10) Evaluate $\lim_{x \to 0} \dfrac{2x}{\sqrt{a+x} - \sqrt{a-x}}$.

(11) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x}$.

(12) Evaluate $\lim_{x \to 0} \dfrac{x}{\sqrt{1+x} - 1}$.

(13) Evaluate $\lim_{x \to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$.

(14) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$, when $f(x) = \sqrt{ax+b}$.

(15) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$, when $f(x) = (mx+c)^n$.

2. Evaluating Limits when $x \to a$

(1) Evaluate $\lim_{x \to 1} \left(6x^2 - 4x + 3\right)$.

(2) Evaluate $\lim_{x \to 7} \dfrac{x^2 - 49}{x - 7}$.

(3) Evaluate $\lim_{x \to 2} \dfrac{x^2 - 6x + 8}{x - 2}$.

(4) Evaluate $\lim_{x \to -5} \dfrac{2x^2 + 9x - 5}{x + 5}$.

(5) Evaluate $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

(6) Evaluate $\lim_{x \to 3} \dfrac{x^2 - 4x + 3}{x^2 - 2x - 3}$.

(7) Evaluate $\lim_{x \to -2} \dfrac{x^3 + 8}{x + 2}$.

(8) Evaluate $\lim_{x \to 3} \dfrac{x^4 - 81}{x - 3}$.

(9) Evaluate $\lim_{x \to 5} \dfrac{x^5 - 3125}{x - 5}$.

(10) Evaluate $\lim_{x \to a} \dfrac{x^{12} - a^{12}}{x - a}$.

(11) Evaluate $\lim_{x \to a} \dfrac{x^{5/2} - a^{5/2}}{x - a}$.

(12) Evaluate $\lim_{x \to a} \dfrac{(x+2)^{5/3} - (a+2)^{5/3}}{x - a}$.

(13) Evaluate $\lim_{x \to 4} \dfrac{x^3 - 64}{x^2 - 16}$.

(14) Evaluate $\lim_{x \to 2} \dfrac{x^5 - 32}{x^3 - 8}$.

(15) Evaluate $\lim_{x \to 1} \dfrac{x^n - 1}{x - 1}$.

(16) Evaluate $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a}$.

(17) Evaluate $\lim_{x \to 2} \dfrac{\sqrt{3 - x} - 1}{2 - x}$.

(18) Evaluate $\lim_{x \to a} \dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$.

(19) Evaluate $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$.

3. Evaluating Limits when $x \to \infty$

(1) Evaluate $\lim_{x \to \infty} \dfrac{x + 2}{x - 2}$.

(2) Evaluate $\lim_{x \to \infty} \dfrac{3x^2 + 2x - 5}{5x^2 + 3x + 1}$.

(3) Evaluate $\lim_{x \to \infty} \dfrac{x^2 - 7x + 11}{3x^2 + 10}$.

(4) Evaluate $\lim_{x \to \infty} \dfrac{2x^3 - 5x + 7}{7x^3 + x^2 - 6}$.

(5) Evaluate $\lim_{x \to \infty} \dfrac{2x^3 - 5x + 7}{7x^3 + x^2 - 6}$.

(6) Evaluate $\lim_{x \to \infty} \frac{(3x-1)(4x-5)}{(x-6)(x-3)}$.

(7) Evaluate $\lim_{n \to \infty} \left(+ \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} \right)$.

(8) Evaluate $\lim_{x \to \infty} \frac{x}{\sqrt{4x^2 - 1} - 1}$.

(9) Evaluate $\lim_{x \to -\infty} 2^x$.

(10) Evaluate $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

(11) Evaluate $\lim_{t \to \infty} \frac{t+1}{t^2+1}$.

(12) Evaluate $\lim_{n \to \infty} \sqrt{n^2 + 1} + n$.

(13) Evaluate $\lim_{n \to \infty} \sqrt{n^2 + n} + n$.

4. Limits with Exponential and Logarithm Functions

(1) Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

(2) Evaluate $\lim_{x \to 0} \frac{a^x - 1}{x}$.

(3) Evaluate $\lim_{x \to 0} \frac{\ln(1+x)}{x}$.

(4) Evaluate $\lim_{x \to 0} (1+x)^{1/x}$.

(5) Evaluate $\lim_{x \to 0} \frac{a^x - b^x}{x}$.

(6) Evaluate $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$.

(7) Evaluate $\lim_{x \to -\infty} 2^x$.

(8) Explain why $\lim_{x \to -1} \ln(x)$ does not exist.

(9) Explain why $\lim_{x \to 0} 2^{1/x}$ does not exist.

(10) Explain why $\lim_{x \to 1} 2^{1/(x-1)}$ does not exist.

(11) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ where $f(x)=e^{\sqrt{x}}$.

(12) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ where $f(x)=\ln(a\,x+b)$.

(13) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ where $f(x)=x^{x}$.

5. Limits with Trigonometric Functions

(1) Evaluate $\lim_{x \to 0} \dfrac{\sin 3x}{4x}$.

(2) Evaluate $\lim_{x \to 0} \dfrac{\sin x \cos x}{3x}$.

(3) Evaluate $\lim_{x \to 0} \dfrac{\tan x}{x}$.

(4) Evaluate $\lim_{x \to 0} \dfrac{1-\cos x}{\sin^2 x}$.

(5) Evaluate $\lim_{x \to 0} \dfrac{\tan a\,x}{\tan b\,x}$.

(6) Evaluate $\lim_{x \to 0} \dfrac{\sin(x/4)}{x}$.

(7) Evaluate $\lim_{x \to 0} \dfrac{\sin m\,x}{\tan n\,x}$.

(8) Evaluate $\lim_{\theta \to 0} \dfrac{1-\cos 6\theta}{\theta}$.

(9) Evaluate $\lim_{x \to 0} \dfrac{1-\cos 2x}{3\tan^2 x}$.

(10) Evaluate $\lim_{x \to 0} \dfrac{\cos^2 x}{1-\sin x}$.

(11) Evaluate $\lim_{x \to 0} \dfrac{\tan 2x - x}{3x-\sin x}$.

(12) Evaluate $\lim_{x \to a} \dfrac{\sin x - \sin a}{x-a}$.

(13) Evaluate $\lim_{x \to 0} \dfrac{\sin 5x - \sin 3x}{\sin x}$.

(14) Evaluate $\lim_{x \to 0} \dfrac{\tan 3x - 2x}{3x-\sin^2 x}$.

(15) Evaluate $\lim_{x \to 0} \frac{x^2 - \tan 2x}{\tan x}$.

(16) Evaluate $\lim_{x \to \pi/4} \frac{1 - \tan x}{x - \pi/4}$.

(17) Evaluate $\lim_{x \to 0} \frac{\tan(x/2)}{3x}$.

(18) Evaluate $\lim_{x \to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$.

(19) Show that if
$\lim_{x \to 0} kx \csc x = \lim_{x \to 0} x \csc kx$,
then
$k = \pm 1$.

(20) Evaluate $\lim_{h \to 0} \frac{\sin(a + h) - \sin a}{h}$.

(21) Evaluate $\lim_{h \to \infty} \frac{\cos(\pi / h)}{h - 2}$.

6. Limits with Inverse Trigonometric Functions

(1) Evaluate $\lim_{x \to 1} \frac{1 - x}{\arccos^2 x}$.

(2) Evaluate $\lim_{x \to 1/\sqrt{2}} \frac{x - \cos(\arcsin x)}{1 - \tan(\arcsin x)}$.

(3) Evaluate $\lim_{x \to 0} \frac{x\left(1 - \sqrt{1 - x^2}\right)}{\arcsin^3(x)\sqrt{1 - x^2}}$.

(4) Evaluate $\lim_{x \to 1} \frac{1 - x}{\pi - 2\arcsin x}$.

(5) Evaluate $\lim_{x \to 1} \frac{\arctan 2x}{\sin 3x}$.

7. Additional Limits

(1) Let $p \in \mathbb{R}_{>0}$. Evaluate $\lim_{n \to \infty} \frac{1}{n^p}$.

(2) Let $r \in \mathbb{R}$ such that $|r| < 1$. Evaluate $\lim_{n \to \infty} r^n$.

(3) Let $a \in \mathbb{R}_{>0}$. Evaluate $\lim_{n \to \infty} a^{\frac{1}{n}}$.

(4) Evaluate $\lim_{n\to\infty} n^{\frac{1}{n}}$.

(5) Let $a \in \mathbb{R}_{>0}$. Evaluate $\lim_{n\to\infty} \frac{a^n}{n!}$.

(6) Evaluate $\lim_{n\to\infty} \frac{\log n}{n^p}$.

(7) Let $a \in \mathbb{R}$. Evaluate $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$.

(8) Let $a,\ p \in \mathbb{R}$ with $a > 1$ and $p > 0$. Evaluate $\lim_{n\to\infty} \frac{n^p}{a^n}$.

(9) Evaluate $\lim_{n\to\infty} \frac{n}{2n+3}$.

(10) Evaluate $\lim_{n\to\infty} \frac{n}{n+1} - \frac{n+1}{n}$.

(11) Evaluate $\lim_{n\to\infty} \frac{1-n}{n^3}$.

(12) Evaluate $\lim_{n\to\infty} \frac{3n-1}{2n+5}$.

(13) Evaluate $\lim_{n\to\infty} \frac{n+1}{\sqrt{n}}$.

(14) Evaluate $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$.

(15) Evaluate $\lim_{n\to\infty} 1 + (-1)^{n+1}$.

(16) Evaluate $\lim_{n\to\infty} n^{(-1)^n}$.

(17) Evaluate $\lim_{n\to\infty} a_n$ when $a_n = \sqrt{3}$.

(18) Evaluate $\lim_{n\to\infty} \frac{n}{2^n}$.

(19) Evaluate $\lim_{n\to\infty} \cos\frac{n\pi}{2}$.

(20) Evaluate $\lim_{n\to\infty} (1 + (-1)^n)\frac{1}{n}$.

(21) Evaluate $\lim_{n\to\infty} \frac{4n^2 - 2n + \cos n}{3n^2 + 7n + 6}$.

(22) Evaluate $\lim_{n\to\infty} \frac{3n-5}{\sqrt{2n^2+1} + \sqrt{3n^2+4}}$.

(23) Evaluate $\lim_{n\to\infty} \dfrac{2n^2+6n+2}{3n^3-n^2-n}$.

(24) Evaluate $\lim_{n\to\infty} \sqrt{n^2+3n} - \sqrt{n^2+4}$.

(25) Evaluate $\lim_{n\to\infty} \sqrt[n]{2n}$.

(26) Evaluate $\lim_{n\to\infty} \left(1-\dfrac{1}{n^2}\right)^n$.

(27) Evaluate $\lim_{n\to\infty} \sqrt[n]{n^2}$.

(28) Evaluate $\lim_{n\to\infty} \dfrac{\log n+1}{n}$.

(29) Evaluate $\lim_{n\to\infty} \dfrac{\log(n+1)}{n}$.

(30) Evaluate $\lim_{n\to\infty} \dfrac{\log n}{\sqrt{n}}$.

(31) Evaluate $\lim_{n\to\infty} \dfrac{n^n}{(n+3)^{n+1}}$.

(32) Evaluate $\lim_{n\to\infty} \dfrac{(2n)^3}{3^n}$.

(33) Evaluate $\lim_{n\to\infty} \left(\dfrac{n}{n+3}\right)^n$.

(34) Evaluate $\lim_{n\to\infty} \dfrac{n^5}{2^n}$.

(35) Evaluate $\lim_{n\to\infty} \dfrac{2^n}{(2n)!}$.

(36) Evaluate $\lim_{n\to\infty} \dfrac{n!}{10^{6n}}$.

(37) Evaluate $\lim_{n\to\infty} \sqrt[n]{2n+1}$.

(38) Evaluate $\lim_{n\to\infty} \sqrt[n]{n^2+n}$.

(39) Evaluate $\lim_{n\to\infty} \dfrac{n!}{n^n}$.

(40) Evaluate $\lim_{n\to\infty} \sin\left(\dfrac{\pi}{3}+\dfrac{1}{n}\right)$.

(41) Evaluate $\lim_{n\to\infty} \tanh n$.

(42) Evaluate $\lim_{n\to\infty} n\sin\left(\frac{\pi}{n}\right)$.

(43) Evaluate $\lim_{n\to\infty} (n+1)\log\left(\frac{n-1}{n}\right)$.

(44) Evaluate $\lim_{n\to\infty} \arctan\left(\cos\left(\frac{\pi}{n}\right)\right)$.

(45) Evaluate $\lim_{n\to\infty} \frac{1}{n}\cosh\left(\log\left(\frac{2n^2+1}{n}\right)\right)$.

(46) Evaluate $\lim_{x\to 0} \frac{1-\frac{1}{2}x^2-\cos x}{x^4}$.

(47) Evaluate $\lim_{x\to 0} \frac{x-\arctan x}{x^3}$.

(48) Evaluate $\lim_{n\to\infty} \frac{3^n-n!}{n!+4^n-2}$.

(49) Evaluate $\lim_{n\to\infty} \frac{\cos n+\log n+n^5}{5n^5-2}$.

(50) Evaluate $\lim_{n\to\infty} \frac{(-1)^n n^{\frac{1}{n}}}{3^{\frac{1}{n}}}$.

(51) Evaluate $\lim_{n\to\infty} n\left(\frac{\pi}{2}-\arctan n\right)$.

(52) Evaluate $\lim_{n\to\infty} \frac{2n^3-5^n+2^n}{5^n-5}$.

(53) Evaluate $\lim_{n\to\infty} \frac{n!+3^n}{2^n+3}$.

(54) Evaluate $\lim_{n\to\infty} \sqrt[n]{4+n}$.

(55) Evaluate $\lim_{n\to\infty} \frac{\log\left(n^2+3\right)}{\log(3n+5)}$.

(56) Evaluate $\lim_{n\to\infty} \log\left(n^2+1\right)-\log\left(n^2+3\right)$.

(57) Evaluate $\lim_{n\to\infty} \frac{n!+3^n}{3^n+\log n}$.

(58) Evaluate $\lim_{n\to\infty} \left(\frac{n-1}{n-1}\right)^n$.

(59) Let $p\in\mathbb{R}_{>0}$. Evaluate $\lim_{x\to\infty} x^p e^{-x}$.

8. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.