Sequences

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1. Sequences

Let Y be a set. A **sequence** $(y_1, y_2, y_3, \dots)$ in Y is a function

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \longrightarrow & Y \\ n & \longmapsto & y_n. \end{array}$$

Let Y be a set with a partial order $\le$. Let $(y_1, y_2, \dots)$ be a sequence in Y. The sequence $(y_1, y_2, \dots)$ is **increasing** if $(y_1, y_2, \dots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then} y_i \le y_{i+1}.$$

The sequence $(y_1, y_2, \dots)$ is **decreasing** if $(y_1, y_2, \dots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then} y_i \ge y_{i+1}.$$

Let Y be a metric space. Let $(y_1, y_2, \dots)$ be a sequence in Y. The sequence is **bounded** if the set

$$\{y_1, y_2, \dots\} \text{ is bounded.}$$

The sequence $(y_1, y_2, \dots)$ is **contractive** if $(y_1, y_2, \dots)$ satisfies: There exists $\alpha \in (0, 1)$ such that

$$\text{if } i \in \mathbb{Z}_{>0} \text{ then } d(y_i, y_{i+1}) \le \alpha d(y_{i-1}, y_i).$$

The sequence $(y_1, y_2, \dots)$ is **Cauchy** if $(y_1, y_2, \dots)$ satisfies

> if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(y_m, y_n) < \varepsilon$.

Let $l \in Y$. The sequence $(y_1, y_2, \dots)$ **converges** to l if $(y_1, y_2, \dots)$ satisfies

> if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(y_n, l) \le \varepsilon$.

Let $(y_1, y_2, \dots)$ be a sequence in $\mathbb{R}$ (or more generally, any totally ordered set with the order topology). The **upper limit** of $(y_1, y_2, \ldots)$ is

$$\limsup y_n = \lim_{n\to\infty} \sup\{y_{\mathrm{n}}, y_{n+1}, \ldots\}.$$

The **lower limit** of $(y_1, y_2, \ldots)$ is

$$\liminf y_n = \lim_{n\to\infty} \inf\{y_{\mathrm{n}}, y_{n+1}, \ldots\}.$$

Example: If $y_n = (-1)^n\left(1 - \frac{1}{n}\right)$ then

$$\limsup y_n = 1 \qquad \text{and} \qquad \liminf y_n = -1.$$

Proposition 1.1 *Let* $(y_1, y_2, \ldots)$ *be a sequence in* $\mathbb{R}$. *Then*

a. $\limsup y_n = \sup\{\text{cluster points of } (y_1, y_2, \ldots)\}$, *and*
b. $\liminf y_n = \inf\{\text{cluster points of } (y_1, y_2, \ldots)\}$.

2. References [PLACEHOLDER]

[BG] A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmanian*, Duke Math. J. **107** no. 3, (2001), 561-575; arXiv:math/9909077v2, MR1828302 (2002e:20083)