15.03.2010 ①

Lecture 7: Limits, Real Analysis with applications

$\lim_{n\to\infty} a_n = l$ means a_n gets closer and closer to l as n gets larger and larger.

$\lim_{x\to a} f(x) = l$ means $f(x)$ gets closer and closer to l as x gets closer and closer to a.

Distance and absolute value

The absolute value on $\mathbb{R}$ is $\mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \longmapsto |x|$ given by

$$|x| = \begin{cases} x, & \text{if } x \in \mathbb{R}_{>0} \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The complex numbers is

$$\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\} \text{ with } i^2 = -1$$

The absolute value on $\mathbb{C}$ is $\mathbb{C} \to \mathbb{R}_{\geq 0}$, $x \longmapsto |x|$ given by

$$|x+iy| = |\sqrt{x^2+y^2}|$$

The vector space $\mathbb{R}^3$ is

$$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$$

The absolute value on $\mathbb{R}^3$ is $\mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, $v = (x,y,z) \longmapsto |v|$ given by

$$|v| = |\sqrt{x^2+y^2+z^2}|, \text{ if } v = (x,y,z).$$

So

$\lim_{n\to\infty} a_n = \ell$ means $\lim_{n\to\infty} |a_n - \ell| = 0$

$\lim_{x\to a} f(x) = \ell$ means $\lim_{x\to a} |f(x) - \ell| = 0$.

Official definitions in Maths.

The sequence (a_n) converges to ℓ if (a_n) satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $|a_n - \ell| < \varepsilon$.

Write $\lim_{n\to\infty} a_n = \ell$ if (a_n) converges to ℓ.

The function $f(x)$ converges to ℓ as $x \to a$ if $f(x)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-a| < \delta$ then $|f(x) - \ell| < \varepsilon$.

Write $\lim_{x\to a} f(x) = \ell$ if $f(x)$ converges to ℓ as $x \to a$.

A function $f(x)$ is continuous at $x=a$ if

$$\lim_{x\to a} f(x) = f(a).$$

MOST IMPORTANT PROPERTY of absolute value:

$$|x+y| \le |x| + |y|.$$

Useful properties of limits

Theorem

(a) Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Then

(a1) $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

(a2) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$,

(a3) $\lim_{n\to\infty} -a_n = -\left(\lim_{n\to\infty} a_n\right)$,

(a4) If (a_n) satisfies: if $n \in \mathbb{Z}_{>0}$ then $a_n \neq 0$,

then $$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$$

(b) Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

Then

(b1) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

(b2) $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$,

(b3) $\lim_{x\to a}(-f(x)) = -\lim_{x\to a} f(x)$,

(b4) If $f(x)$ satsifies: $f(x) \neq 0$ for all x close to a

then $$\lim_{x\to a}\left(\frac{1}{f(x)}\right) = \frac{1}{\lim_{x\to a} f(x)}.$$

Theorem

(a) Assume that (a_n) and (b_n) are sequences in $\mathbb{R}$ and $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

If $a_n \le b_n$ then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

(a') Assume that $f(x)$ and $g(x)$ are real valued functions and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

If $f(x) \le g(x)$ then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

(b) Assume that $\lim_{x\to a} g(x) = \ell$ and $\lim_{y\to \ell} f(y)$ exists. Then $\lim_{y\to\ell} f(y) = \lim_{x\to a} f(g(x))$.

Theorem Let $x \in \mathbb{C}$.

(a) $$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|>1 \\ 1, & \text{if } x=1 \\ \text{diverges}, & \text{if } |x|=1 \text{ and } x\neq 1 \end{cases}$$

(b) Let $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{C}$
$\lim_{x\to a} x^n = a^n$ (i.e. $f(x)=x^n$ is continuous).

(c) Let $a \in \mathbb{C}$.
$\lim_{x\to a} e^x = e^a$ (i.e. $f(x)=e^x$ is continuous).

(d) $$\lim_{n\to\infty} 1+x+x^2+\cdots+x^n = \begin{cases} \frac{1}{1-x}, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|\ge 1 \end{cases}$$

(e) $\lim_{n\to\infty}\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)$ exists.