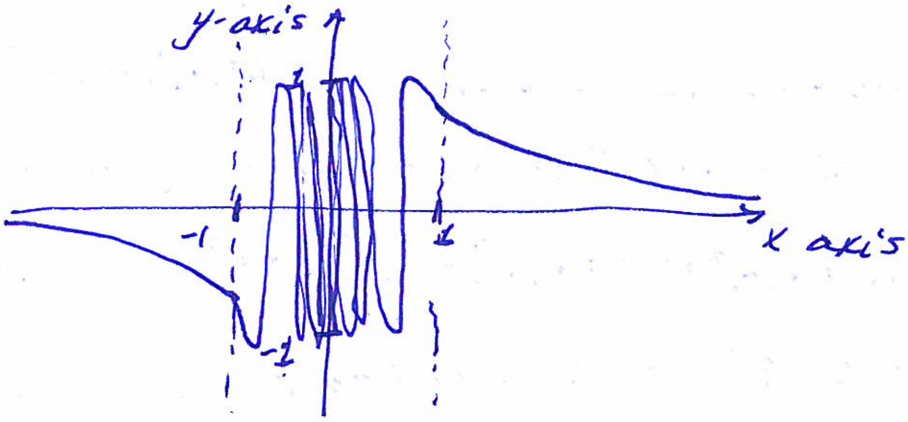
620-295 Real analysis with applications

Lecture 6: Graphing and limits, 11 March 2010

Example Graph $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$



$\lim_{x \to a} f(x) = l$ if $f(x)$ gets closer and closer to l as x gets closer and closer to a.

$\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist because $\sin(\frac{1}{x})$ oscillates between 0 and 1 as x gets closer and closer to 0.

The function $f(x)$ is continuous at $x = a$ if $f(x)$ is such that

$$\lim_{x \to a} f(x) = f(a)$$

The function $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

is not continuous at $x = 0$ because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin(\tfrac{1}{x}) \text{ does not exist.}$$

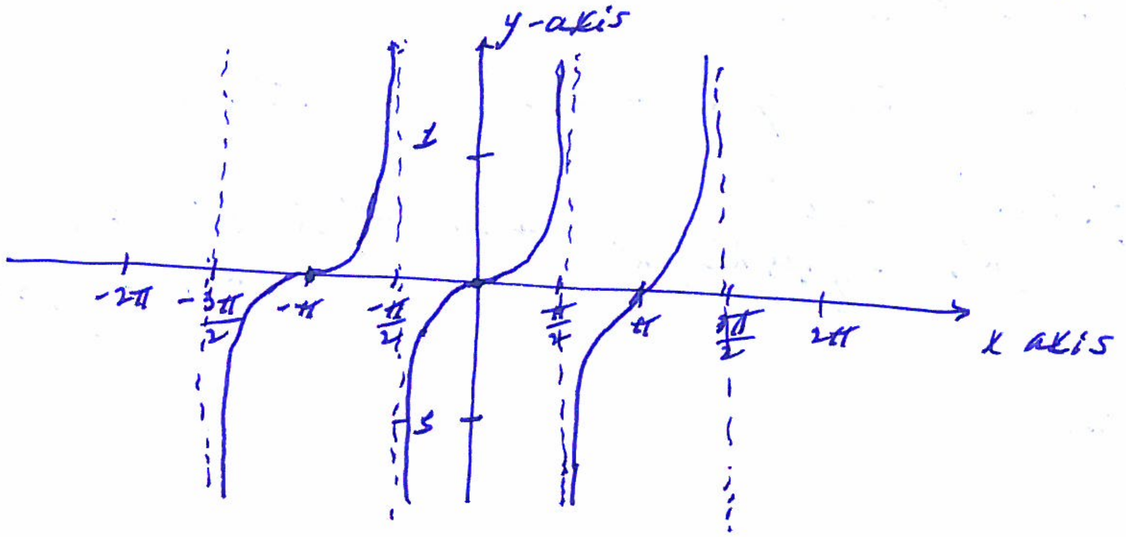
Example Graph $y = \tan x$.

$y = \tan x = \dfrac{\sin x}{\cos x}$.

Notes

(a) If $\sin x = 0$ then $y = 0$

(b) If $\cos x = 0$ then y is very large.

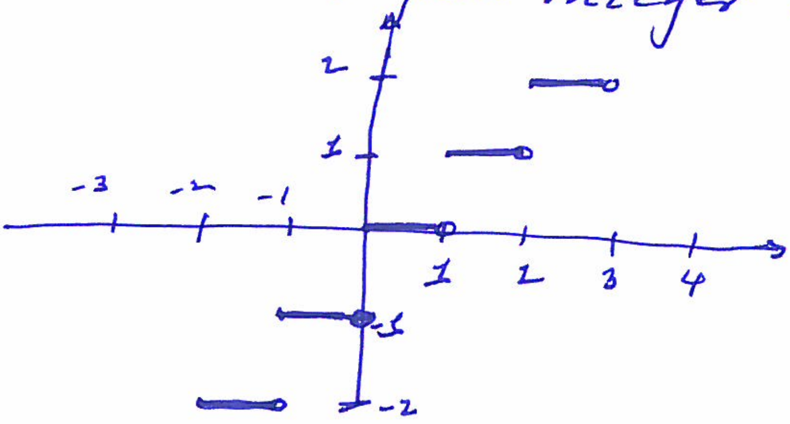


$\lim_{x \to \frac{\pi}{2}} \tan x$ does not exist because $\tan x$ gets larger and larger as x gets closer and closer to $\frac{\pi}{2}$.

Example Graph $y = \lfloor x \rfloor$.

$\lfloor x \rfloor = n$, if $n \in \mathbb{Z}$ and $n \le x < n+1$,

i.e. $\lfloor x \rfloor$ is the largest integer $\le x$.



$\lim_{x\to 1} \lfloor x \rfloor$ does not exist because

$$\lim_{x\to 1^-} \lfloor x \rfloor = 0 \text{ and } \lim_{x\to 1^+} \lfloor x \rfloor = 1,$$

where $\lim_{x\to 1^-} \lfloor x \rfloor$ is the limit of $\lfloor x \rfloor$ as x gets closer and closer to 1 from the <u>negative</u> <u>side</u> of 1,

and $\lim_{x\to 1^+} \lfloor x \rfloor$ is the limit of $\lfloor x \rfloor$ as x gets closer and closer to 1 from the <u>positive</u> <u>side</u> of 1.

The function $f(x) = \lfloor x \rfloor$, for $x \in \mathbb{R}$, is continuous for $x \notin \mathbb{Z}$.

<u>In English</u>: A function $f(x)$ is continuous at $x=a$ if $f(x)$ doesn't jump at $x=a$.

<u>In math</u>: A function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

<u>Example</u> Let $z = \frac{1}{2}e^{i\pi/4}$. Graph the sequence $a_n = z^n$.

A <u>sequence</u> in $\mathbb{C}$ is a function

$$\begin{aligned} \mathbb{Z}_{\geq 0} &\to \mathbb{C} \\ n &\mapsto a_n \end{aligned}$$

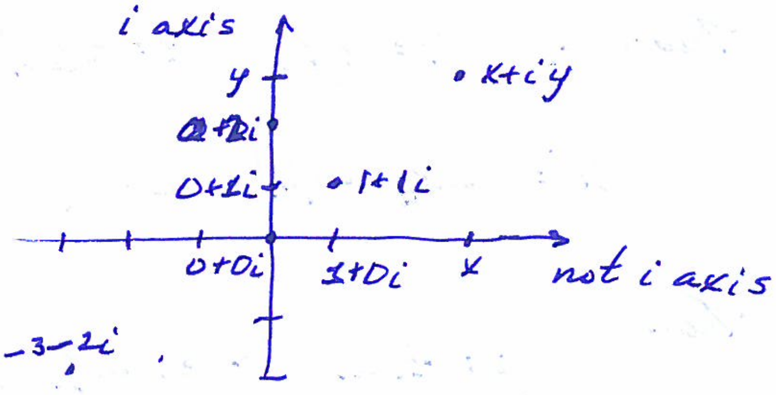
The first 7 terms of this sequence are

$a_1 = \frac{1}{2} e^{i\pi/4}$, $a_2 = \frac{1}{4} e^{i2\pi/4}$, $a_3 = \frac{1}{8} e^{i3\pi/4}$,

$a_4 = \frac{1}{16} e^{i\frac{4\pi}{4}}$, $a_5 = \frac{1}{32} e^{i5\pi/4}$, $a_6 = \frac{1}{64} e^{i6\pi/4}$,

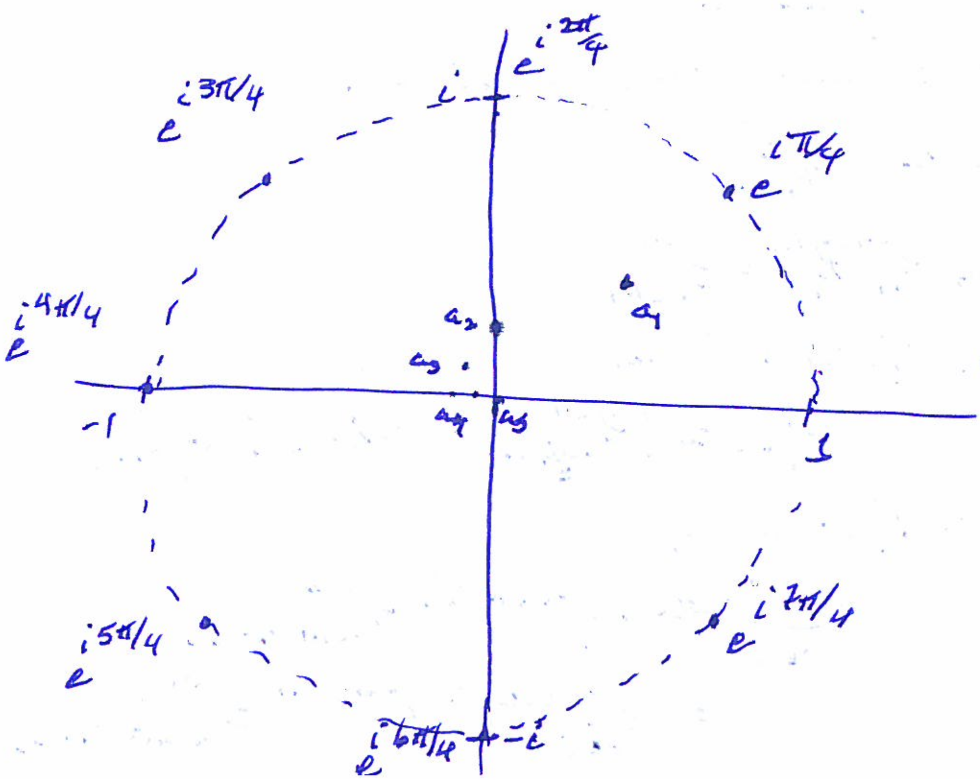
$a_7 = \frac{1}{128} e^{i7\pi/4}$.

Graph $x+iy$ as



So $a_1 = \frac{1}{2} e^{i\pi/4} = \frac{1}{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$a_2 = \frac{1}{4} e^{i\frac{2\pi}{4}} = \frac{1}{4} e^{i\frac{\pi}{2}} = \frac{1}{4}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \frac{1}{2}(0+i)$



$\lim_{n\to\infty} a_n = l$ means a_n gets closer and closer to l as n gets larger and larger.

In our example the a_n are getting closer and closer to $0+0i$, in a spiral.

<u>Definition</u> Let $z = x+iy$ be in $\mathbb{C}$. The <u>absolute</u> <u>value</u> of z is

$$|z| = \sqrt{x^2+y^2}.$$

In English: $|z|$ is the distance from z to $0+0i$.

The <u>complex</u> <u>numbers</u> is

$$\mathbb{C} = \{x+iy \mid x,y \in \mathbb{R}\} \text{ with } i^2=-1.$$