620-295 Real Analysis with applications 08.03.2010 ①

Lecture 4 Induction, sequences, bounds.

Example Prove that

if $n \in \mathbb{Z}_{>0}$ then $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Proof Proof by induction.

Base case: Assume $n=1$.

To show: $1 = \frac{1(1+1)}{2}$.

Righthand side: $\frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$.

Induction step: Let $N \in \mathbb{Z}_{>0}$.

Assume that if $n \in \mathbb{Z}_{>0}$ and $n < N$ then

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

To show: $1+2+\cdots+(N-1)+N = \frac{N(N+1)}{2}$.

Left hand side:

$$1+2+\cdots+(N-1)+N = (1+2+\cdots+(N-1))+N$$

$$= \frac{(N-1)(N-1+1)}{2} + N,$$ by the induction hypothesis,

$$= \frac{(N-1)N}{2} + \frac{2N}{2}$$

$$= \frac{N(N-1+2)}{2} = \frac{N(N+1)}{2}. \quad //$$

Sum notation

Example: Write out the first 10 terms of

$$\sum_{n=4}^{100} (-1)^{n+1} \frac{2^n x^n}{(n-3)^2}$$

Solution:

$$\sum_{n=4}^{100} (-1)^{n+1} \frac{2^n x^n}{(n-3)^2} = (-1)^5 \frac{2^4 x^4}{(4-3)^2} + (-1)^6 \frac{2^5 x^5}{(5-3)^2} + \dots + (-1)^{14} \frac{2^{13} x^{13}}{(13-3)^2} + \dots$$

$$= -2^4 x^4 + \frac{2^5 x^5}{2^2} - \frac{2^6 x^6}{3^2} + \frac{2^7 x^7}{4^2} - \frac{2^8 x^8}{5^2} + \frac{2^9 x^9}{6^2}$$

$$- \frac{2^{10} x^{10}}{7^2} + \frac{2^{11} x^{11}}{8^2} + \frac{2^{12} x^{12}}{9^2} - \frac{2^{13} x^{13}}{10^2} + \dots \quad //$$

Example Write $\sin x$ as a series in sum notation.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$= \frac{x^{2\cdot 0+1}}{(2\cdot 0+1)!} + (-1)^1 \frac{x^{2\cdot 1+1}}{(2\cdot 1+1)!} + (-1)^2 \frac{x^{2\cdot 2+1}}{(2\cdot 2+1)!} + (-1)^3 \frac{x^{2\cdot 3+1}}{(2\cdot 3+1)!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n \in \mathbb{Z}_{\geq 0}} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Bounds and intervals

$\mathbb{R}_{\geq 0}$ is the set of decimal expansions.

$\mathbb{R}$ is the set of positive and negative decimal expansions.

Example: $0.9999\ldots \in \mathbb{R}$, $1.000\ldots \in \mathbb{R}$

$-0.9999\ldots \in \mathbb{R}$, $-1.000\ldots \in \mathbb{R}$.

Let $a, b \in \mathbb{R}$. The sets

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

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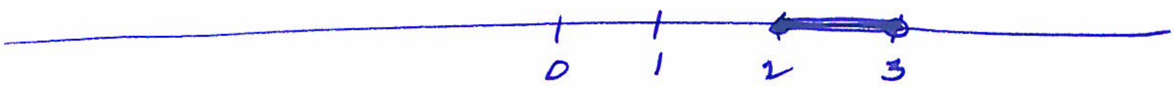
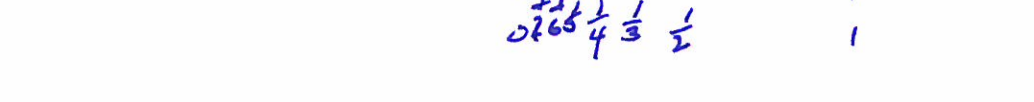
are intervals in $\mathbb{R}$. The set (a,b) is an open interval in $\mathbb{R}$.

Example Graph $[2,3)$.



Example Graph $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$



Let S be a subset of $\mathbb{R}$.

An <u>upper bound</u> of S in $\mathbb{R}$ is $b \in \mathbb{R}$ such that if $x \in S$ then $x \leq b$.

A <u>least</u> upper <u>bound</u> of S in $\mathbb{R}$ is $\sup(S) \in \mathbb{R}$ such that

(a) $\sup(S)$ is an upper bound of S in $\mathbb{R}$,

(b) If b is an upper bound of S in $\mathbb{R}$ then $b \geq \sup(S)$.

A <u>maximum</u> of S is $m \in S$ such that if $x \in \mathbb{R}$ and $x > m$ then $x \notin S$.

<u>Example</u> Write $S = \{x \in \mathbb{R} \mid |x-2| < 3 \text{ and } |x+1| < 1\}$

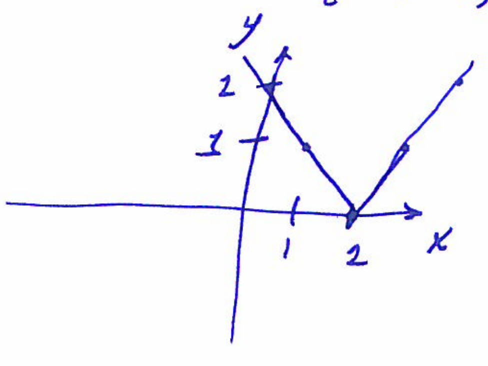
If $x \in \mathbb{R}$ then <u>absolute</u> <u>value</u> of x is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

So
$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases} = \begin{cases} x-2, & \text{if } x \geq 2 \\ 2-x, & \text{if } x < 2. \end{cases}$$

The graph of

$y = x-2$ is



$S_1 = \{x \in \mathbb{R} \mid |x-2| < 3\} = (-1,5)$

$S_2 = \{x \in \mathbb{R} \mid |x+1| < 1\}$

$= \{x \in \mathbb{R} \mid (x+1 \geq 0 \text{ and } x+1 < 1) \text{ or } (x+1 < 0 \text{ and } -(x+1) < 1)\}$

$= \{x \in \mathbb{R} \mid (x \geq -1 \text{ and } x < 0) \text{ or } (x < -1 \text{ and } x+1 > -1)\}$

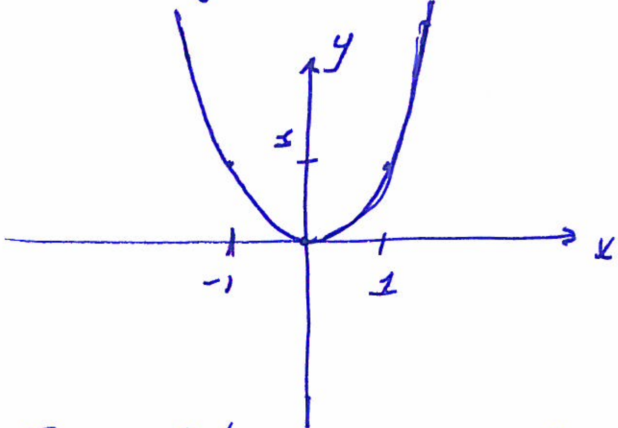
$= \{x \in \mathbb{R} \mid x \in [-1,0) \text{ or } (x < -1 \text{ and } x > -2)\}$

$= \{x \in \mathbb{R} \mid x \in [-1,0) \text{ or } x \in (-2,-1)\}$

$= [-1,0) \cup (-2,-1) = (-2,0).$

<u>Example</u> Graph $S = \{x \in \mathbb{R} \mid x^2 < 9\}$

The graph of $y = x^2$ is



So $S = \{x \in \mathbb{R} \mid x > -3 \text{ and } x < 3\}$

$= (-3,3).$