Lecture 3: Series 620-295 Real Analysis

The set of polynomials is

$$\mathbb{Q}[x] = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_\ell x^\ell \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \text{ and} \\ a_1, \ldots, a_\ell \in \mathbb{Q} \end{array} \right\}$$

The set of series is

$$\mathbb{Q}[[x]] = \{ a_0 + a_1 x + a_2 x^2 + \cdots \mid a_1, a_2, \ldots \in \mathbb{Q} \}.$$

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p(x), q(x) \in \mathbb{Q}[x],\ q(x) \neq 0 \right\}$$

with $\dfrac{p(x)}{q(x)} = \dfrac{r(x)}{s(x)}$ if $p(x)s(x) = r(x)q(x)$.

$$\mathbb{Q}((x)) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p(x), q(x) \in \mathbb{Q}[[x]],\ q(x) \neq 0 \right\}$$

with $\dfrac{p(x)}{q(x)} = \dfrac{r(x)}{s(x)}$ if $p(x)s(x) = r(x)q(x)$.

Examples

$$3x^2 + 2x + 7 \in \mathbb{Q}[x], \qquad \frac{3x^2 + 2x + 7}{(x-1)(x-2)} \in \mathbb{Q}(x).$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \in \mathbb{Q}[[x]], \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \in \mathbb{Q}[[x]]$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} \in \mathbb{Q}((x)).$$

Example Find a series expansion of $\frac{1}{1-x}$.

Another way to say the same thing is:

Write $\frac{1}{1-x}$ as an element of $\mathbb{Q}[[x]]$.

Last lecture we found: $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$

Example Find a series expansion of $\frac{1}{1+x}$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots$$

$$= 1-x+x^2-x^3+x^4-x^5+\cdots$$

Example Find the Maclaurin series for $\log(1+x)$.

Another way to say the same thing is:

Find the Taylor series for $\log(1+x)$ at $x=0$

or Find a series expansion of $\log(1+x)$

or Write $\log(1+x)$ as an element of $\mathbb{Q}[[x]]$

Proof

$$\int \frac{1}{1+x}\,dx = \log(1+x), \text{ since } \frac{d}{dx}\log(1+x) = \frac{1}{1+x}.$$

So

$$\log(1+x) = \int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+x^4+\cdots)\,dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Example Find the Taylor series for $\log x$ at $x=1$.

Another way to say the same thing is

Find a series representation for $\log x$ in powers of $x-1$.

Proof Let $y = x-1$. Then

$$\log x = \log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad //.$$

Example Find the Taylor series for e^x at the point $a=-3$.

i.e. Write $e^x = c_0 + c_1(x-(-3)) + c_2(x-(-3)) + \cdots$

i.e. Write $e^x = c_0 + c_1(x+3) + c_2(x+3)^2 + c_3(x+3)^3 + \cdots$

Well

$$c_0 = e^x\Big|_{x=-3} = e^{-3}.$$

and since

$$\frac{de^x}{dx} = 0 + c_1 + 2c_2(x+3) + 3c_3(x+3)^2 + 4c_4(x+3)^3 + \cdots$$

then

$$c_1 = \frac{de^x}{dx}\Big|_{x=-3} = e^x\Big|_{x=-3} = e^{-3}.$$

Then

$$\frac{d^2e^x}{dx^2} = 2c_2 + 3\cdot 2c_3(x+3) + 4\cdot 3c_4(x+3)^2 + 5\cdot 4c_5(x+3)^3 + \cdots$$

So $\quad 2c_2 = \left.\dfrac{d^2 e^x}{dx^2}\right|_{x=-3} = \left. e^x \right|_{x=-3} = e^{-3}.$

So $\quad c_2 = \dfrac{1}{2} e^{-3}.$

Next $\quad \dfrac{d^3 e^x}{dx^3} = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4 (x+3) + 5\cdot 4\cdot 3\, c_5 (x+3)^2 + 6\cdot 5\cdot 4\, c_6 (x+3)^3 + \ldots$

So $\quad 3\cdot 2 c_3 = \left.\dfrac{d^3 e^x}{dx^3}\right|_{x=-3} = \left. e^x \right|_{x=-3} = e^{-3}.$

So $\quad c_3 = \dfrac{1}{3\cdot 2} e^{-3}.$

Next $\quad \dfrac{d^4 e^x}{dx^4} = 4\cdot 3\cdot 2\, c_4 + 5\cdot 4\cdot 3\cdot 2\, c_5 (x+3) + 6\cdot 5\cdot 4\cdot 3\, c_6 (x+3)^2 + 7\cdot 6\cdot 5\cdot 4\, c_7 (x+3)^4 + \ldots$

So $\quad 4\cdot 3\cdot 2 c_4 = \left.\dfrac{d^4 e^x}{dx^4}\right|_{x=-3} = \left. e^x \right|_{x=-3} = e^{-3}.$

So $\quad c_4 = \dfrac{1}{4\cdot 3\cdot 2} e^{-3}.$

So

$$e^x = c_0 + c_1 (x+3) + c_2 (x+3)^2 + c_3 (x+3)^3 + \ldots$$

$$= e^{-3} + e^{-3}(x+3) + \frac{1}{2} e^{-3}(x+3)^2 + \frac{1}{3\cdot 2} e^{-3}(x+3)^3 + \frac{1}{4\cdot 3\cdot 2} e^{-3}(x+3)^4 + \ldots$$