620-295 Lecture 2 - Expressions 3 March 2000

Real Analysis with applications

Definition: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Example: Prove that $e^{x+y} = e^x e^y$

$$e^{x+y} = 1 + x + y + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \frac{(x+y)^4}{4!} + \cdots$$

$$\begin{aligned} = \; & 1 \\ & + x + y \\ & + \tfrac{1}{2!}(x^2 + 2xy + y^2) \\ & + \tfrac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3) \\ & + \tfrac{1}{4!}(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4) \\ & + \cdots \end{aligned}$$

$$\begin{aligned} = \; & 1 \\ & + x + y \\ & + \tfrac{1}{2!}x^2 + xy + \tfrac{1}{2!}y^2 \\ & + \tfrac{1}{3!}x^3 + \tfrac{1}{2!}x^2y + \tfrac{1}{2!}xy^2 + \tfrac{1}{3!}y^3 \\ & + \tfrac{1}{4!}x^4 + \tfrac{1}{3!}x^3y + \tfrac{1}{2!}x^2\tfrac{1}{2!}y^2 + x\tfrac{1}{3!}y^3 + \tfrac{1}{4!}y^4 \\ & + \tfrac{1}{5!}x^5 + \tfrac{1}{4!}x^4y + \tfrac{1}{3!}x^3\tfrac{1}{2!}y^2 + \tfrac{1}{2!}x^2\tfrac{1}{3!}y^3 + x\tfrac{1}{4!}y^4 + \tfrac{1}{5!}y^5 \\ & + \cdots \end{aligned}$$

$$= e^x + e^x y + e^x\frac{1}{2!}y^2 + e^x\frac{1}{3!}y^3 + e^x\frac{1}{4!}y^4 + \cdots = e^x e^y.$$

Definitions:

$\log x$ is the expression that undoes e^x:

$\log(e^x) = x$ and $e^{\log x} = x$

$\sqrt{x}$ is the expression that undoes x^2

$\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$.

and

$\arcsin(\sin x) = x$ and $\sin(\arcsin x) = x$

$\arccos(\cos x) = x$ and $\cos(\arccos x) = x$

$\tan^{-1}(\tan x) = x$ and $\tan(\tan^{-1} x) = x$.

$\sinh^{-1}(\sinh x) = x$ and $\sinh(\sinh^{-1} x) = x$

$\text{arccosh}(\cosh x) = x$ and $\cosh(\text{arccosh}\, x) = x$.

Note: $\arcsin x$ and $\sin^{-1} x$ have the same meaning.

$\sin^{-1} x$ does not mean $\frac{1}{\sin x}$ (which would be written $(\sin x)^{-1}$).

Derivatives - the definition

(a) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$.

(b) $\frac{d(cf)}{dx} = c\frac{df}{dx}$ if c is a constant.

(c) $\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g$

(d) $\frac{dx}{dx} = 1$ and (e) $\frac{df(g)}{dx} = \frac{df}{dg}\frac{dg}{dx}$.

Consequences:

Example Prove that $\frac{dx^2}{dx} = 2x$.

$$\frac{dx^2}{dx} = \frac{d(x\cdot x)}{dx} = x\frac{dx}{dx} + \frac{dx}{dx}\cdot x = x\cdot 1 + 1\cdot x = 2x.$$

Example Prove that $\frac{de^x}{dx} = e^x$.

Suppose we know that

if $n \in \mathbb{Z}_{>0}$ then $\frac{dx^n}{dx} = nx^{n-1}$.

Then

$$\frac{de^x}{dx} = \frac{d\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)}{dx}$$

$$= 0 + 1 + \frac{1}{2!}2x + \frac{1}{3!}3x^2 + \frac{1}{4!}4x^3 + \frac{1}{5!}5x^4 + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x.$$

Example Prove that $\frac{d\log x}{dx} = \frac{1}{x}$.

Proof.

$$1 = \frac{dx}{dx} = \frac{d\, e^{(\log x)}}{dx}$$

$$= \frac{d\, e^{\log x}}{d\log x}\cdot\frac{d\log x}{dx}$$

$$= e^{\log x}\cdot\frac{d\log x}{dx} = x\,\frac{d\log x}{dx}.$$

So $\frac{d\log x}{dx} = \frac{1}{x}$

Example Prove that $\dfrac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$

Proof

$$(1+x+x^2+x^3+\cdots)(1-x) = \begin{aligned}&1+x+x^2+x^3+x^4+\cdots\\ &\quad -x-x^2-x^3-x^4-\cdots\end{aligned}$$

$$= 1.$$

Divide both sides by $1-x$.

So $1+x+x^2+x^3+\cdots = \dfrac{1}{1-x}$.

Example Prove that $\dfrac{1}{1+x} = 1-x+x^2-x^3+x^4+\cdots$

Proof $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots$

$$= 1-x+x^2-x^3+x^4-x^5+\cdots$$

Example Prove that $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

Proof Since $\dfrac{d\log(1+x)}{dx} = \dfrac{1}{1+x}$, then $\displaystyle\int \frac{1}{1+x}\,dx = \log(1+x)$.

So

$$\log(1+x) = \int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+x^4-x^5+\cdots)\,dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

<u>Example</u> Prove that $e^{ix} = \cos x + i\sin x$.
(By definition $i^2=-1$).

<u>Proof</u>

$$\cos x + i\sin x = \frac{e^{ix}+e^{-ix}}{2} + i\,\frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{e^{ix}+e^{-ix}+e^{ix}-e^{-ix}}{2} = \frac{2e^{ix}}{2} = e^{ix}. //$$

<u>Example</u> Prove that $\sin 2x = 2\sin x\cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$.

<u>Proof</u>

$$\cos 2x + i\sin 2x = e^{i2x} = e^{i(x+x)}$$

$$= e^{ix}e^{ix} = (\cos x + i\sin x)(\cos x + i\sin x)$$

$$= \cos^2 x + i^2\sin^2 x + 2i\sin x\cos x$$

$$= (\cos^2 x - \sin^2 x) + i\,(2\sin x\cos x).$$

So $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2\sin x\cos x$. //