Derivatives of trig functions

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Define

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots, \end{aligned}$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Example: Find $\dfrac{de^x}{dx}$.

$$\begin{aligned} \frac{de^x}{dx} &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right) \\ &= 0 + 1 + \frac{1}{2!}2x + \frac{1}{3!}3x^2 + \frac{1}{4!}4x^3 + \frac{1}{5!}5x^4 + \frac{1}{6!}6x^5 + \frac{1}{7!}7x^6 + \cdots \\ &= 1 + \frac{1}{2}2x + \frac{1}{3\cdot 2!}3x^2 + \frac{1}{4\cdot 3!}4x^3 + \frac{1}{5\cdot 4!}5x^4 + \frac{1}{6\cdot 5!}6x^5 + \frac{1}{7\cdot 6!}7x^6 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\ &= e^x. \end{aligned}$$

Example: Find $\dfrac{d\sin x}{dx}$.

$$\begin{aligned}\frac{d\sin x}{dx} &= \frac{d}{dx}\big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots\big)\\ &= 1 - \frac{1}{3!}3x^2 + \frac{1}{5!}5x^4 - \frac{1}{7!}7x^6 + \frac{1}{9!}9x^8 - \frac{1}{11!}11x^{10} + \frac{1}{13!}13x^{12} - \cdots\\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \cdots\\ &= \cos x.\end{aligned}$$

Example: Find $\dfrac{d\cos x}{dx}$.

$$\begin{aligned}\frac{d\cos x}{dx} &= \frac{d}{dx}\big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots\big)\\ &= 0 - \frac{1}{2!}2x + \frac{1}{4!}4x^3 - \frac{1}{6!}6x^5 + \frac{1}{8!}8x^7 - \frac{1}{10!}10x^9 + \frac{1}{12!}12x^11 - \cdots\\ &= -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \frac{1}{9!}x^9 + \frac{1}{11!}1x^11 - \cdots\\ &= -\big(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}1x^11 + \cdots\big)\\ &= -\sin x.\end{aligned}$$

Example: Find $\dfrac{d\tan x}{dx}$.

$$\begin{aligned}\frac{d\tan x}{dx} &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\left(\sin x(\cos x)^{-1}\right)\\ &= \sin x\frac{d(\cos x)^{-1}}{dx} + \frac{d\sin x}{dx}(\cos x)^{-1}\\ &= \sin x(-1)(\cos x)^{-2}\frac{d\cos x}{dx} + \cos x\cdot\frac{1}{\cos x}\\ &= -\frac{\sin x}{\cos^2 x}(-\sin x) + 1 = \frac{\sin^2 x}{\cos^2 x} + 1\\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

Example: Find $\dfrac{d\sec x}{dx}$.

$$\begin{aligned}\frac{d\sec x}{dx} &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left((\cos x)^{-1}\right) = (-1)(\cos x)^{-2}\frac{d\cos x}{dx}\\ &= -\frac{1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x\sec x.\end{aligned}$$

Example: Find $\dfrac{d\csc x}{dx}$.

$$\frac{d\csc x}{dx} = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}\left((\sin x)^{-1}\right) = (-1)(\sin x)^{-2}\frac{d\sin x}{dx}$$
$$= -\frac{1}{\sin^2 x}(\cos x) = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = -\cot x\csc x.$$

Example: Find $\frac{d\cot x}{dx}$.

$$\begin{aligned}\frac{d\cot x}{dx} &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{d}{dx}\left(\cos x(\sin x)^{-1}\right)\\ &= \cos x\frac{d(\sin x)^{-1}}{dx} + \frac{d\cos x}{dx}(\sin x)^{-1}\\ &= \cos x(-1)(\sin x)^{-2}\frac{d\sin x}{dx} + -(\sin x)\cdot\frac{1}{\sin x}\\ &= -\frac{\cos x}{\sin^2 x}\cdot\cos x - 1 = \frac{-\cos^2 x}{\sin^2 x} - 1\\ &= \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = \csc^2 x.\end{aligned}$$

Example: Find $\frac{d\ \sinh x}{dx}$.

$$\begin{aligned}\frac{d\ \sinh x}{dx} &= \frac{d}{dx}\big(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots\big)\\ &= 1 + \frac{1}{3!}3x^2 + \frac{1}{5!}5x^4 + \frac{1}{7!}7x^6 + \frac{1}{9!}9x^8 + \frac{1}{11!}11x^{10} + \frac{1}{13!}13x^{12} + \cdots\\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} + \cdots\\ &= \cosh x.\end{aligned}$$

Example: Find $\frac{d\ \cosh x}{dx}$.

$$\begin{aligned}\frac{d\ \cosh x}{dx} &= \frac{d}{dx}\big(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots\big)\\ &= 0 + \frac{1}{2!}2x + \frac{1}{4!}4x^3 + \frac{1}{6!}6x^5 + \frac{1}{8!}8x^7 + \frac{1}{10!}10x^9 + \frac{1}{12!}12x^11 + \cdots\\ &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{1}{11!}1x^11 + \cdots\\ &= \sinh x.\end{aligned}$$

Example: Find $\frac{d\ \tanh x}{dx}$.

$$\begin{aligned}
\frac{d\ \tanh x}{dx} &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{d}{dx}\left(\sinh x(\cosh x)^{-1}\right) \\
&= \sinh x \frac{d(\cosh x)^{-1}}{dx} + \frac{d \sinh x}{dx}(\cosh x)^{-1} \\
&= \sinh x(-1)(\cosh x)^{-2}\frac{d\cosh x}{dx} + \cosh x \cdot \frac{1}{\cosh x} \\
&= -\frac{\sinh x}{\cosh^2 x}\cdot \sinh x + 1 = -\frac{\sinh^2 x}{\cosh^2 x} + 1 \\
&= \frac{-\sinh^2 x + \cosh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x.
\end{aligned}$$

Example: Find $\dfrac{d\ \mathrm{sech} x}{dx}$.

$$\begin{aligned}
\frac{d\ \mathrm{sech} x}{dx} &= \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = \frac{d}{dx}\left((\cosh x)^{-1}\right) = (-1)(\cosh x)^{-2}\frac{d\cosh x}{dx} \\
&= -\frac{1}{\cosh^2 x}\cdot \sinh x = -\frac{\sinh x}{\cosh^2 x} = -\frac{\sinh x}{\cosh x}\cdot\frac{1}{\cosh x} = -\tanh x \,\mathrm{sech} x.
\end{aligned}$$

Example: Find $\dfrac{d\ \mathrm{csch} x}{dx}$.

$$\begin{aligned}
\frac{d\ \mathrm{csch} x}{dx} &= \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{d}{dx}\left((\sinh x)^{-1}\right) = (-1)(\sinh x)^{-2}\frac{d\sinh x}{dx} \\
&= -\frac{1}{\sinh^2 x}(\cosh x) = -\frac{\cosh x}{\sinh^2 x} = -\frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x} = -\coth x \,\mathrm{csch} x.
\end{aligned}$$

Example: Find $\dfrac{d\ \coth x}{dx}$.

$$\begin{aligned}
\frac{d\ \coth x}{dx} &= \frac{d}{dx}\left(\frac{1}{\tanh x}\right) = \frac{d(\tanh x)^{-1}}{dx} = (-1)(\tanh x)^{-2}\frac{d\tanh x}{dx} \\
&= -\frac{1}{\tanh^2 x}\frac{d\tanh x}{dx} = -\frac{1}{\tanh^2 x}\cdot \mathrm{sech}^2 x \\
&= -\frac{1}{\frac{\sinh^2 x}{\cosh^2 x}}\cdot\frac{1}{\cosh^2 x} = -\frac{1}{\sinh^2 x} = -\mathrm{csch}^2 x.
\end{aligned}$$