

Chapter 0. SETS AND FUNCTIONS

The basic building blocks of mathematics are sets and functions. Functions allow us to compare sets.

§1T. Sets

(0.1.1) Definition.

- A **set** is a collection of objects which are called **elements**. We write $s \in S$ if s is an element of a set S .
- The **emptyset**, \emptyset , is the set with no elements.
- A **subset** T of a set S is a set T such that if $t \in T$ then $t \in S$. We write $T \subseteq S$.
- Two sets S and T are **equal** if $S \subseteq T$ and $T \subseteq S$. We write $T = S$.
- Let S and T be sets. The **union** of S and T is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$.

$$S \cup T = \{u \mid u \in S \text{ or } u \in T\}.$$

- Let S and T be sets. The **intersection** of S and T is the set $S \cap T$ of all u such that $u \in S$ and $u \in T$.

$$S \cap T = \{u \mid u \in S \text{ and } u \in T\}.$$

- Let S and T be sets. S and T are **disjoint** if $S \cap T = \emptyset$.
- Let S and T be sets. S is a **proper subset** of T if $S \subseteq T$ and $S \neq T$. We write $S \subsetneq T$.
- The **product** of two sets S and T is the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$,

$$S \times T = \{(s, t) \mid s \in S, t \in T\}.$$

More generally, given sets S_1, \dots, S_n , the **product** $\prod_i S_i$ is the set of all tuples (s_1, \dots, s_n) such that $s_i \in S_i$.

- The elements of a set S are **indexed** by the elements of a set I if each element of S is labeled by a unique element of I . If $i \in I$, s_i denotes the corresponding element of S .

We will use the following notations:

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integers.

$\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers.

$\mathbb{P} = \{1, 2, \dots\}$ is the set of positive integers.

$[1, n] = \{1, 2, \dots, n\}$ for each $n \in \mathbb{P}$.

$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{P}\}$ is the set of rational numbers.

\mathbb{R} is the set of real numbers.

\mathbb{C} is the set of complex numbers.

Example. Let S, T, U , and V be the sets $S = \{1, 2\}$, $U = \{1, 2\}$, $T = \{1, 2, 3\}$, and $V = \{2, 3\}$. Then

a) $S \subseteq U \subseteq T$. d) $U \cap V = \{2\}$.

b) $U \not\subseteq V$. e) $S \times T = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$.

c) $U \cup V = T$.

HW: Show that the emptyset is a subset of every set.

§2T. Functions

(2.2.1) Definition.

- Let S and T be sets. A **map** or **function** $f: S \rightarrow T$ is given by associating to each element $s \in S$ a unique element $f(s) \in T$.

$$\begin{aligned} f: S &\rightarrow T \\ s &\mapsto f(s). \end{aligned}$$

- Often in mathematics one will try to define a function without being exactly sure if what has been defined really is a function. In order to check that a function is **well defined** one must check that
 - For every $s \in S$, $f(s) \in T$.
 - If $s_1 = s_2$ then $f(s_1) = f(s_2)$.
- Let S and T be sets. Two functions $f: S \rightarrow T$ and $g: S \rightarrow T$ are **equal** if

$$f(s) = g(s), \quad \text{for all } s \in S.$$

We write $f = g$.

- Let S and T be sets and let $f: S \rightarrow T$ be a function. Let $R \subseteq S$. The **restriction** of f to R is the function $f|_R$ given by

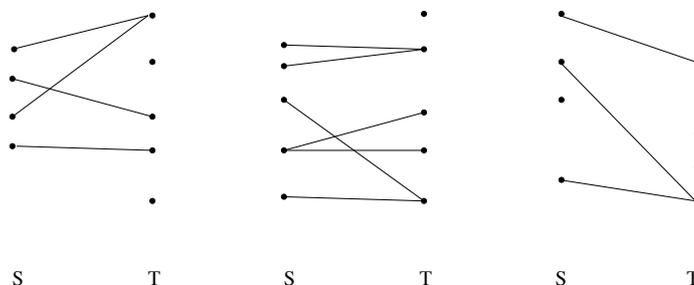
$$\begin{aligned} f|_R: R &\rightarrow T \\ r &\mapsto f(r). \end{aligned}$$

- A map $f: S \rightarrow T$ is **injective** or **one-to-one** if it satisfies

$$\text{If } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \text{ then } s_1 = s_2.$$

- A map $f: S \rightarrow T$ is **surjective** or **onto** if for each element $t \in T$ there exists $s \in S$ such that $f(s) = t$.
- A map is **bijective** if it is both injective and surjective.

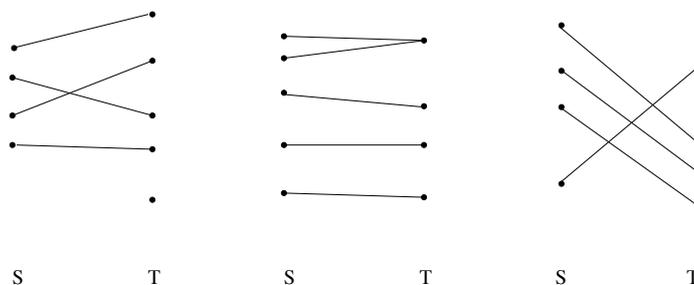
Examples. It is useful to visualize a function $f: S \rightarrow T$ as a graph with edges $(s, f(s))$ connecting elements of $s \in S$ and $f(s) \in T$. With this idea in mind we have the following.



a) function

b) not a function

c) not a function



d) injective function

e) surjective function

f) bijective function

In these pictures we are viewing the elements of the left column as elements of the set S and the elements of the right column as the elements of a set T . In order to be a function the graph must have exactly one edge adjacent to each element of S . A function is injective if there is at most one edge adjacent to each point of T . A function is surjective if there is at least one edge adjacent to each point of T .

Composition of Functions

(2.2.2) Definition.

- Let $f: S \rightarrow T$ and $g: T \rightarrow U$ be functions. The **composition** of f and g is the function $g \circ f$ given by

$$(g \circ f): S \rightarrow U \\ s \mapsto g(f(s)).$$

- Let S be a set. The **identity map** on a set S is the map given by

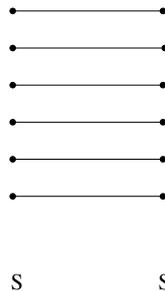
$$\iota_S: S \rightarrow S \\ s \mapsto s.$$

- Let $f: S \rightarrow T$ be a function. An **inverse function** to f is a function $f^{-1}: T \rightarrow S$ such that

$$f \circ f^{-1} = \iota_T \quad \text{and} \\ f^{-1} \circ f = \iota_S$$

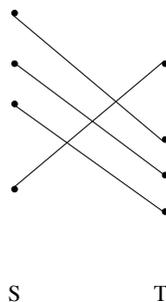
where ι_T and ι_S are the identity functions on T and S respectively.

If we visualize functions as graphs, the identity function ι_S looks something like



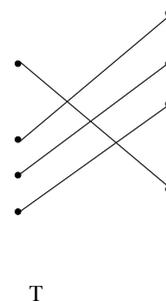
The function ι_S

In the pictures below, if the left graph is a pictorial representation of a function $f: S \rightarrow T$ then the inverse function to f , $f^{-1}: T \rightarrow S$, is represented by the graph on the right.



S T

f



T S

f^{-1}

(2.2.3) Proposition. Let $f: S \rightarrow T$ be a function. An inverse function to f exists if and only if f is bijective.

Pictorially, the graph, below left, represents a function $g: S \rightarrow T$ which is not bijective. The inverse function to g does not exist in this case; the graph of a possible candidate (below right) is not the graph of a function.



Operations

(2.2.4) Definition.

- An **operation** on a set S is a map $\circ: S \times S \rightarrow S$. If $s_1, s_2 \in S$ we write $s_1 \circ s_2$ instead of $\circ((s_1, s_2))$.
- An operation on a set S is **associative** if, for all $s_1, s_2, s_3 \in S$,

$$(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3).$$

- An operation on a set S is **commutative** if, for all $s_1, s_2 \in S$,

$$s_1 \circ s_2 = s_2 \circ s_1.$$

Example. The map $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$\begin{aligned} +: \mathbb{Z} \times \mathbb{Z} &\rightarrow \mathbb{Z} \\ (i, j) &\mapsto i + j \end{aligned}$$

is an operation. This operation is both commutative and associative.

The map $-: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ given by

$$\begin{aligned} -: \mathbb{Z} \times \mathbb{Z} &\rightarrow \mathbb{Z} \\ (i, j) &\mapsto i - j \end{aligned}$$

is an operation. This operation is noncommutative and nonassociative.

Relations

(2.2.5) Definition.

- A **relation** on a set S is a subset of $S \times S$. We write $s_1 \sim s_2$ if the pair (s_1, s_2) is in this subset.
- A relation is **reflexive** if, for each $s \in S$,

$$s \sim s.$$

- A relation is **symmetric** if

$$s_1 \sim s_2 \iff s_2 \sim s_1.$$

- A relation is **transitive** if

$$s_1 \sim s_2 \text{ and } s_2 \sim s_3 \implies s_1 \sim s_3.$$

- An **equivalence relation** on a set S is a relation on S that is reflexive, symmetric and transitive.

Example. Let S be the set $\{1, 2, 6\}$. Then:

- $R_1\{(1, 1), (2, 6), (6, 1)\}$ is a relation on S .
- R_1 is not reflexive, not symmetric, and not transitive.
- $R_2 = \{(1, 1), (2, 6), (6, 1), (2, 1)\}$ is a relation on S .
- R_2 is transitive but not symmetric and not reflexive.

(2.2.6) Definition.

- Let S be a set and let \sim be an equivalence relation on S . The **equivalence class** of an element $s \in S$ is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

- A **partition** of a set S is a collection of subsets S_α such that:

- If $s \in S$ then $s \in S_\alpha$ for some S_α .
- If $S_\alpha \cap S_\beta \neq \emptyset$ then $S_\alpha = S_\beta$.

(2.2.7) Proposition.

- Let S be a set and let \sim be an equivalence relation on S . The set of equivalence classes of the relation \sim is a partition of S .
- Let S be a set and let $\{S_\alpha\}$ be a partition of S . Then the relation defined by

$$s \sim t \text{ if } s \text{ and } t \text{ are in the same } S_\alpha$$

is an equivalence relation on S .

Proposition 2.2.7 shows that the concepts of an equivalence relation on S and of a partition of S are essentially the same. Each equivalence relation on S determines a partition on S and vice versa.

Example. Let $S = \{1, 2, 3, \dots, 10\}$. Let \sim be the equivalence relation determined by

$$\begin{array}{lll} 1 \sim 5, & 2 \sim 3, & 9 \sim 10, \\ 1 \sim 7, & 5 \sim 8, & 10 \sim 4. \end{array}$$

Since we are requiring that \sim is an equivalence relation, we are assuming that we have all the other relations we need so that \sim is reflexive, symmetric, and transitive:

$$\begin{array}{l} 1 \sim 1, 2 \sim 2, \dots, 10 \sim 10, \\ 5 \sim 7, 7 \sim 8, 7 \sim 5, 5 \sim 1, \dots \end{array}$$

Then the equivalence classes are given by

$$\begin{aligned} [1] &= [5] = [7] = [8] = \{1, 5, 7, 8\} \\ [2] &= [3] = \{2, 3\} \\ [6] &= \{6\} \\ [4] &= [9] = [10] = \{4, 9, 10\}, \end{aligned}$$

and the sets

$$S_1 = \{1, 5, 7, 8\}, S_2 = \{2, 3\}, S_3 = \{6\}, \text{ and } S_4 = \{4, 9, 10\}$$

form a partition of S .

Cardinality of Sets

How big is a set?

(2.2.8) Definition.

- Let S and T be sets. S and T have the **same cardinality**, $\text{Card}(S) = \text{Card}(T)$, if there is a bijective map from S to T .

Notation: Let S be a set. Then

$$\text{Card}(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ n & \text{if } \text{Card}(S) = \text{Card}(\{1, 2, \dots, b\}); \\ \infty & \text{otherwise.} \end{cases}$$

Note: Even if $\text{Card}(S) = \infty$ and $\text{Card}(T) = \infty$, one may have that $\text{Card}(S) \neq \text{Card}(T)$.

(2.2.9) Definition.

- A set S is **finite** if $\text{Card}(S) \neq \infty$.
- A set S is **infinite** if S is not finite.
- A set S is **countable** if either S is finite or if $\text{Card}(S) = \text{Card}(\mathbb{P})$.
- A set S is **countably infinite** if S is countable and not finite.
- A set S is **uncountable** if S is not countable.

HW: Show that $\text{Card}(\mathbb{R}) = \infty$ and $\text{Card}(\mathbb{Q}) = \infty$ and that $\text{Card}(\mathbb{R}) \neq \text{Card}(\mathbb{Q})$.