Math 521: Lecture 7

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1 Fields of fractions

Let A be a commutative ring. A **zero divisor** is an element $a \in A$ such that there exists $b \in A$ such that $b \neq 0$ and $ab = 0$.

A **integral domain** is a commutative ring A with no zero divisors except 0.

Let A be an integral domain. A **field of fractions** of A is the set

$$\mathbb{F} = \left\{ \frac{a}{b} \mid a, b \in A, b \neq 0 \right\},$$

with

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if } ad = bc,$$

and operations given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{cd}.$$

Theorem 1.1. *Let A be an integral domain. Let $\mathbb{F}$ be the field of fractions of A.*

(a) The operations on $\mathbb{F}$ are well defined and $\mathbb{F}$ is a field.

(b) The map

$$\begin{array}{rccc} \iota\colon & A & \longrightarrow & \mathbb{F} \\ & a & \longmapsto & \frac{a}{1} \end{array}$$

is an injective ring homomorphism.

(c) If $\mathbb{K}$ is a field with an injective ring homomorphism $\zeta\colon A \to \mathbb{K}$ then there is a unique ring homomorphism $\varphi\colon \mathbb{F} \to \mathbb{K}$ such that $\zeta = \varphi \circ \iota$.