Math 521: Lecture 2

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1 Sets and functions

The basic building blocks of mathematics are sets and functions. Functions are for comparing sets.

Sets.

A **set** is a collection of **elements**. Write $s \in S$ if s is an element of a set S.

The **emptyset** $\emptyset$ is the set with no elements.

A **subset** T of a set S is a set T such that if $t \in T$ then $t \in S$. Write $T \subseteq S$ if T is a subset of S.

Two sets S and T are **equal** if $S \subseteq T$ and $T \subseteq S$. Write $T = S$ if S and T are equal sets.

Let S and T be sets. S is a **proper subset** of T if $S \subseteq T$ and $S \neq T$. Write $S \underset{\neq}{\subset} T$ if S is a proper subset of T.

Let S be a set and let A be a subset of S. The **complement** of A in S is the set

$$A^c = \{b \in S \mid b \notin A\}.$$

Let S and T be sets. The **union** of S and T is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$.

$$S \cup T = \{u \mid u \in S \text{ or } u \in T\}.$$

Let S and T be sets. The **intersection** of S and T is the set $S \cap T$ of all u such that $u \in S$ and $u \in T$.

$$S \cap T = \{u \mid u \in S \text{ and } u \in T\}.$$

Let S and T be sets. The sets S and T are **disjoint** if $S \cap T = \emptyset$.

The **product** of two sets S and T is the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$,

$$S \times T = \{(s, t) \mid s \in S, t \in T\}.$$

More generally, given sets $S_1, \ldots, S_n$, the **product** $\prod_i S_i$ is the set of all tuples $(s_1, \ldots, s_n)$ such that $s_i \in S_i$.

The elements of a set S are **indexed** by the elements of a set I if each element of S is labeled by a unique element of I. If $i \in I$, s_i denotes the corresponding element of S.

Example. Let S, T, U, and V be the sets $S = \{1, 2\}$, $U = \{1, 2\}$, $T = \{1, 2, 3\}$, and $V = \{2, 3\}$. Then

(a) $S \subseteq U \subseteq T$.

(b) $U \not\subseteq V$.

(c) $U \cup V = T$.

(d) $U \cap V = \{2\}$.

(e) $S \times T = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$.

Functions.

Let S and T be sets. A **function** or **map** $f\colon S \to T$ is given by associating to each element $s \in S$ an element $f(s) \in T$.

$$\begin{aligned} f : S &\to T \\ s &\mapsto f(s) \end{aligned}$$

Often in mathematics one will try to define a function without being exactly sure if what has been defined really is a function. In order to check that a function is **well defined** one must check that

(a) If $s \in S$ then $f(s) \in T$.

(b) If $s_1 = s_2$ then $f(s_1) = f(s_2)$.

Let S and T be sets. Two functions $f\colon S \to T$ and $g\colon S \to T$ are **equal** if

$$f(s) = g(s), \quad \text{for all } s \in S.$$

Write $f = g$ if f and g are equal functions.

Let $f\colon S \to T$ be a function. Let $K \subseteq S$. The **image** of K is the set

$$f(K) = \{f(k) \mid k \in K\}.$$

Let $f\colon S \to T$ be a function. Let $L \subseteq T$. The **inverse image** of L is the set

$$f^{-1}(L) = \{s \in S \mid f(s) \in L\}.$$

Let $f\colon S \to T$ be a function. The **image** of f is the set $f(S)$.

Let $f\colon S \to T$ be a function and let $t \in T$. The **fiber** of f over t is the set $f^{-1}(t)$.

A function $f\colon S \to T$ is **injective** if it satisfies the condition

$$\text{If } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \text{ then } s_1 = s_2.$$

A map $f\colon S \to T$ is **surjective** if it satisfies the condition

$$\text{if } t \in T \text{ then there exists } s \in S \text{ such that } f(s) = t.$$

A function is **bijective** if it is both injective and surjective.

Examples. It is useful to visualize a function $f\colon S \to T$ as a graph with edges $\big(s, f(s)\big)$ connecting elements of $s \in S$ and $f(s) \in T$. With this idea in mind we have the following.

PICTURE

In these pictures we are viewing the elements of the left column as elements of the set S and the elements of the right column as the elements of a set T. In order to be a function the graph must have exactly one edge adjacent to each element of S. A function is injective if there is at most one edge adjacent to each point of T. A function is surjective if there is at least one edge adjacent to each point of T.

Let $f\colon S \to T$ be a function and let $R \subseteq S$. The **restriction** of f to R is the function $f|_R$ given by

$$\begin{array}{rcl} f|_R\colon R & \to & T \\ r & \mapsto & f(r). \end{array}$$

Let S be a set, let R be a subset of S and let $f\colon R \to T$ be a function. An **extension** of f to S is a function $g\colon S \to T$ such that

$$\text{if} \quad r \in R \quad \text{then} \quad g(r) = f(r).$$

Composition of functions.

Let $f\colon S \to T$ and $g\colon T \to U$ be functions. The **composition** of f and g is the function $g \circ f$ given by

$$\begin{array}{rcl} (g \circ f)\colon S & \to & U \\ s & \mapsto & g\big(f(s)\big). \end{array}$$

Let S be a set. The **identity map** on a set S is the map given by

$$\begin{array}{rcl} \mathrm{id}_S\colon S & \to & S \\ s & \mapsto & s. \end{array}$$

Let $f\colon S \to T$ be a function. An **inverse function** to f is a function $f^{-1}\colon T \to S$ such that

$$f \circ f^{-1} = \mathrm{id}_T \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{id}_S.$$

where id_T and id_S are the identity functions on T and S respectively.

If we visualize functions as graphs, the identity function id_S looks something like

PICTURE

In the pictures below, if the left graph is a pictorial representation of a function $f\colon S \to T$ then the inverse function to f, $f^{-1}\colon T \to S$, is represented by the graph on the right.

PICTURE

Proposition 1. *Let $f\colon S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Pictorially, the graph, below left, represents a function $g\colon S \to T$ which is not bijective. The inverse function to g does not exist in this case; the graph of a possible candidate (below right) is not the graph of a function.

PICTURE

Let $f\colon S \to T$ be a surjective function. A **section** of f is a function $s\colon T \to S$ such that $f \circ s = \mathrm{id}_T$.

Let $f\colon S \to T$ be an injective function. A **retraction** of f is a function $f\colon S \to T$ such that $r \circ f = \mathrm{id}_S$.