Math 521: Lecture 16

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1 Sequences

Let X be a set. A **sequence** $(x_1, x_2, x_3, \ldots)$ of points in X is a function

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \longrightarrow & X \\ n & \longmapsto & x_n \end{array}$$

Let X be a set and let $(x_1, x_2, \ldots)$ be a sequence in X. A **limit** of the sequence $(x_1, x_2, \ldots)$ is a limit point of the sequence with respect to the Fréchet filter on $\mathbb{Z}_{>0}$. Write

$$y = \lim_{n\to\infty} x_n \qquad \text{if } y \text{ is a limit of the sequence } (x_1, x_2, \ldots).$$

The sequence $(x_1, x_2, \ldots)$ **converges** if $\lim_{n\to\infty} x_n$ exists and is unique.

The sequence $(x_1, x_2, \ldots)$ **diverges** if it does not converge.

Let X be a totally ordered set and let $(x_1, x_2, \ldots)$ be a sequence in X. The **upper limit** $\limsup x_n$ of the sequence $(x_1, x_2, \ldots)$ is the supremum of the set $\{x_1, x_2, \ldots\}$,

$$\limsup x_n = \sup(\{x_1, x_2, \ldots\}).$$

Let X be a totally ordered set and let $(x_1, x_2, \ldots)$ be a sequence in X. The **lower limit** $\liminf x_n$ of the sequence $(x_1, x_2, \ldots)$ is the infimum of the set $\{x_1, x_2, \ldots\}$,

$$\liminf x_n = \inf(\{x_1, x_2, \ldots\}).$$

A sequence $(x_1, x_2, \ldots)$ is **bounded** if the set $\{x_1, x_2, \ldots\}$ is bounded.

A sequence $(x_1, x_2, \ldots)$ is **monotonically increasing** if it is such that if $i \in \mathbb{Z}_{>0}$ then $x_i \le x_{i+1}$.

A sequence $(x_1, x_2, \ldots)$ is **monotonically decreasing** if it is such that if $i \in \mathbb{Z}_{>0}$ then $x_i \ge x_{i+1}$.

2 Series

Let X be an abelian group and let $(a_1, a_2, \ldots)$ be a sequence in X. The **series** $\sum_{n=1}^{\infty} a_n$ is

$$\text{the sequence} \qquad (s_1, s_2, s_3, \ldots), \qquad \text{where} \qquad s_k = s_1 + s_2 + \cdots + s_k.$$

Write

$$\sum_{n=1}^{\infty} a_n = a \qquad \text{if} \qquad \lim_{n\to\infty} s_n = a.$$

The series $\sum_{n=1}^{\infty} a_n$ **converges** if the sequence $(s_1, s_2, \ldots)$ converges.

The series $\sum_{n=1}^{\infty} a_n$ **diverges** if the sequence $(s_1, s_2, \ldots)$ diverges.

The series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 2.1. *Suppose that* $\sum_{n=1}^{\infty} a_n = a$ *and* $\sum_{n\in\mathbb{Z}_{>0}} a_n$ *converges absolutely. Then*

(a) Every rearrangement of $\sum_{n=1}^{\infty} a_n$ *converges to* a.

(b) If $\sum_{n=1}^{\infty} b_n$ *is a series and* $\sum_{n=1}^{\infty} b_n = b$ *then*

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = ab.$$