Math 521: Lecture 15

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1 Neighborhoods

Let X be a topological space and let $x \in X$. A **neighborhood** of x is a subset N of X such that there exists an open subset U of X with $x \in U$ and $U \subseteq N$.

Let X be a topological space and let $E \subset X$. A **neighborhood** of E is a subset N of X such that there exists an open subset U of X with $E \subseteq U \subseteq N$.

2 Continuous functions

Continuous functions are for comparing topological spaces.

Let X and Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if it satisfies the condition

$$\text{if } V \text{ is an open subset of } Y \text{ then } f^{-1}(V) \text{ is an open subset of } X.$$

Let X and Y be topological spaces. Let $a \in X$. A function $f\colon X \to Y$ is **continuous** at a if it satisfies the condition

$$\text{if } V \text{ is a neighborhood of } f(a) \text{ in } Y \text{ then } f^{-1}(V) \text{ is a neighborhood of } a \text{ in } X.$$

Theorem 2.1. *Let X and Y be topological spaces and let $a \in X$. A function $f\colon X \to Y$ is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.*

3 Filters

Let X be a set. A **filter** on X is a collection $\mathcal{F}$ of subsets of X such that

1. (a) if $E \subseteq X$ such that there exists $U \in \mathcal{F}$ with $E \supseteq U$ then $E \in \mathcal{F}$,
2. (b) finite intersections of elements of F are in $\mathcal{F}$,
3. (c) $\emptyset \notin \mathcal{F}$.

Let X be a set and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be filters on X. The filter $\mathcal{F}_1$ is **finer** than $\mathcal{F}_2$ is $\mathcal{F}_1 \supseteq \mathcal{F}_2$.

Let X be a topological space and let $x \in X$. The **neighborhood filter** of x is the collection

$$\mathcal{F} = \{\text{neighborhoods of } x.\}$$

The **Fréchet filter** on $\mathbb{Z}_{>0}$ is the collection

$$\mathcal{F} = \{\text{complements of finite subsets of } \mathbb{Z}_{>0}\}.$$

Let $\mathcal{F}$ be a filter on a set X. A **filter base** of $\mathcal{F}$ is a collection $\mathcal{B}$ of subsets of X such that

$$\mathcal{F} = \{\text{subsets of } X \text{ that contain a set in } \mathcal{B}\}.$$

Let $\mathcal{F}$ be a filter on a set X. A **subbase** of $\mathcal{F}$ is a collection $\mathcal{S}$ of subsets of X such that

$$\mathcal{B} = \{\text{finite intersections of elements of } \mathcal{S}\}$$

is a base of the filter $\mathcal{F}$.

4 Limits points and cluster points

Let X be a set and let $\mathcal{F}$ be a filter on X. A **limit point** of $\mathcal{F}$ is a point $x \in X$ such that the neighborhood filter of x is finer than $\mathcal{F}$.

Let X be a set and let $\mathcal{B}$ be a filter base of a filter $\mathcal{F}$ on X. A **cluster point** of $\mathcal{B}$ is a point $x \in X$ such that x is in the closure of each set in $\mathcal{B}$.

Let X be a set with a filter $\mathcal{F}$ and let Y be a topological space. Let $f\colon X \to Y$ be a function.

A **limit point** of $f\colon X \to Y$ is a limit point of the filter base $f(\mathcal{F})$. Write

$$\text{Write} \quad y = \lim_{\mathcal{F}} f(x) \quad \text{if } y \text{ is a limit point of } f.$$

A **cluster point** of $f\colon X \to Y$ is a cluster point of the filter base $f(\mathcal{F})$.

Let X be a set. A **sequence** $(x_1, x_2, x_3, \ldots)$ of points in X is a function

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \longrightarrow & X \\ n & \longmapsto & x_n \end{array}$$

Let X be a set and let $(x_1, x_2, \ldots)$ be a sequence in X. A **limit** of the sequence $(x_1, x_2, \ldots)$ is a limit point of the sequence with respect to the Fréchet filter on $\mathbb{Z}_{>0}$. Write

$$y = \lim_{n\to\infty} f(x)$$

if y is a limit of the sequence $(x_1, x_2, \ldots)$.

Let X be a set and let $(x_1, x_2, \ldots)$ be a sequence in X. A **cluster point** of the sequence $(x_1, x_2, \ldots)$ is a cluster point of the sequence with respect to the Fréchet filter on $\mathbb{Z}_{>0}$.

Let X and Y be topological spaces. Let $a \in X$. A **limit of** $f(x)$ **as** x **approaches** a is a limit point of f with respect to the neighborhood filter of a. Write

$$y = \lim_{x \to a} f(x),$$

if y is a limit of $f(x)$ as x approaches a.

Let X and Y be topological spaces and let $a \in X$. Let $f \colon X \to Y$ be a function. The function f is **continuous at** a if it satifies the condition,

$$\text{if } N \text{ is a neighborhood of } f(a) \text{ in } Y \quad \text{then } f^{-1} \text{ is a neighborhood of } a \text{ in } X.$$

Theorem 4.1. *Let X and Y be topological spaces and let $a \in X$. A function $f \colon X \to Y$ is continuous at a if and only if* $\lim_{x \to a} f(x) = f(a)$.

5 Compact sets

Let X be a set. An **ultrafilter** on X is a filter $\mathcal{F}$ such that there is no filter on X which is strictly finer that $\mathcal{F}$.

Let X be a topological space. The space X is **quasicompact** if every filter on X has a cluster point.

Theorem 5.1. *Let X be a topological space. The following are equivalent.*

1. *(a) Every filter on X has at least one cluster point.*
2. *(b) Every ultrafilter on X is convergent.*
3. *(c) Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.*
4. *(d) Every open cover of X contains a finite subcover.*

A topological space is **Hausdorff** if any two distinct points of X have disjoint neighborhoods.

A topological space is **compact** if it is quasicompact and Hausdorff.