Math 521: Lecture 12

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1 Ordered fields

An **ordered monoid** is a commutative monoid G with an ordering $\le$ such that

$$\text{if } x, y, z \in G \text{ and } x \le y \text{ then } x + z \le y + z.$$

An **ordered group** is an abelian group G with an ordering $\le$ such that

$$\text{if } x, y, z \in G \text{ and } x \le y \text{ then } x + z \le y + z.$$

An **ordered ring** is a commutative ring A with an ordering $\le$ such that

(a) A is an ordered group under $+$,

(b) If $x, y \in A$ and $x \ge 0$ and $y \ge 0$ the $xy \ge 0$.

An **ordered field** is a field $\mathbb{F}$ with a total ordering $\le$ such that $\mathbb{F}$ is an ordered ring.

Let G be an ordered group and let $x \in G$. The element x is **positive** if $x \ge 0$. The element x is **negative** if $x \le 0$. The element x is **strictly positive** if $x > 0$. The element x is **strictly negative** if $x < 0$.

Let G be a lattice ordered group. If $x \in G$ define

$$x^+ = \sup(x, 0), \qquad \text{and} \qquad x^- = \sup(-x, 0).$$

Let G be a lattice ordered group and let $x \in G$. The **absolute value** of x is

$$|x| = \sup(x, -x).$$

Let G be an ordered group. Let $x, y \in G$. The elements x and y are **coprime** if $\inf(x, y) = 0$.

Let G be an ordered group. Let $x \in G$. The element is **irreducible** if it is a minimal element of the set of strictly positive elements of G.

Let $\mathbb{F}$ be an ordered field. If $x \in \mathbb{F}$ define

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The nonnegative integers $\mathbb{Z}_{\geq 0}$ with the ordering defined by

$$x \leq y \quad \text{if there is an } n \in \mathbb{Z}_{\geq 0} \text{ with } y = x + n,$$

is an ordered monoid. There is a unique extension of this ordering to $\mathbb{Z}$ so that $\mathbb{Z}$ is an ordered group. There is a unique extension of this ordering to $\mathbb{Q}$ so that $\mathbb{Q}$ is an ordered field. Show that

$$\text{if } x, y \in \mathbb{Q} \text{ then } x \leq y \text{ if and only if } y - x \geq 0.$$

We still need the proper characterization of $\mathbb{R}$ as an ordered field that contains $\mathbb{Q}$ and satisfies the least upper bound property. What is the proper uniqueness statement? Should we put Dedekind cuts here?