620-295 Real Analysis with applications

Problem Sheet 3

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1. Cardinality

1. Define the following and give an example for each:
   (a) cardinality,
   (b) finite,
   (c) infinite,
   (d) countable,
   (e) uncountable.

2. Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

3. Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z})$.

4. Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Q})$.

5. Show that $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$.

6. Show that $\mathrm{Card}(\mathbb{C}) = \mathrm{Card}(\mathbb{R})$.

7. Let S be a set. Show that $\mathrm{Card}(S) = \mathrm{Card}(S)$.

8. Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ then $\mathrm{Card}(T) = \mathrm{Card}(S)$.

9. Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ and $\mathrm{Card}(T) = \mathrm{Card}(U)$ then $\mathrm{Card}(S) = \mathrm{Card}(U)$.

10. Define $\mathrm{Card}(S) \leq \mathrm{Card}(T)$ if there exists an injective function $f : S \to T$. Show that if Card $(S) \leq \mathrm{Card}(T)$ and $\mathrm{Card}(T) \leq \mathrm{Card}(S)$ then $\mathrm{Card}(S) = \mathrm{Card}(T)$.

2. Sequences

1. Define the following and give an example for each:
   (a) sequence,
   (b) converges (for a sequence),
   (c) diverges (for a sequence),
   (d) limit (of a sequence),
   (e) sup (of a sequence),
   (f) inf (of a sequence),
   (g) lim sup (of a sequence),
   (h) lim inf (of a sequence),
   (i) bounded (for a sequence),
   (j) increasing (for a sequence),
   (k) decreasing (for a sequence),
   (l) monotone (for a sequence),
   (m) Cauchy sequence,
   (m) contractive sequence,

2. Prove that if (a_n) converges then $\lim_{n\to\infty} a_n$ is unique.

3. Prove that if (a_n) converges then (a_n) is bounded.

4. Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n + b_n = a + b$.

5. Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n b_n = ab$.

6. Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and $b_n \neq 0$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

7. Prove that if $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} c_n = \ell$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} b_n = \ell$.

8. Prove that if (a_n) is increasing and bounded above then (a_n) converges.

9. Prove that if (a_n) is increasing and not bounded above then (a_n) diverges.

10. Prove that if (a_n) is decreasing and bounded below then (a_n) converges.

11. Prove that if (a_n) is decreasing and not bounded below then (a_n) diverges.

12. Prove that every sequence (a_n) of real numbers has a monotonic subsequence.

13. (Bolzano-Weirstrass) Prove that every sequence (a_n) of real or complex numbers has a convergent subsequence.

14. Prove that every Cauchy sequence (a_n) of real or complex numbers converges.

15. Prove that every convergent sequence (a_n) is a Cauchy sequence.

16. Graph and determine the sup, inf, lim sup, lim inf and convergence of the following sequences:
    (a) $a_n = (\text{-}1)^n$,
    (b) $a_n = \frac{1}{n}$,
    (c) $a_n = \frac{(n!)^2 5^n}{(2n)!}$,
    (d) $a_1 = 3, a_n = \frac{1}{2}\left(a_{n-1} + \frac{5}{a_{n-1}}\right)$,
    (e) $a_n = \left(1 + \frac{1}{n}\right)^n$,
    (f) $a_n = e^{in\pi/7}$,

17. Does the sequence given by $\frac{n}{2n+1}$ converge? If so, what is the limit?

18. Does the sequence given by $\sqrt{n}$ converge? If so, what is the limit?

19. Does the sequence given by $\frac{1}{\sqrt{n}}$ converge? If so, what is the limit?

20. Does the sequence given by $\sqrt{n+1} - \sqrt{n}$ converge? If so, what is the limit?

21. Does the sequence given by $\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$ converge? If so, what is the limit?

22. Does the sequence given by $\frac{n}{n^2+1}$ converge? If so, what is the limit?

23. Does the sequence given by $\frac{2n}{n+1}$ converge? If so, what is the limit?

24. Does the sequence given by $\frac{3n+1}{2n+5}$ converge? If so, what is the limit?

25. Does the sequence given by $\frac{n^2-1}{2n^2+3}$ converge? If so, what is the limit?

26. Show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above by 3.

27. Let $a \in \mathbb{R}$ with $|a| < 1$. Does the sequence given by a^n converge? If so, what is the limit?

28. Let $a \in \mathbb{R}$ with $a > 0$. Does the sequence given by $a^{1/n}$ converge? If so, what is the limit?

29. Does the sequence given by $n^{1/n}$ converge? If so, what is the limit?

30. Let $a \in \mathbb{R}$ with $a > 0$. Fix a positive real number x_1. Let $x_{n+1} = \frac{1}{2}(x_n + a / x_n)$. Show that the sequence x_n converges to $\sqrt{a}$.

31. Let $\alpha, \beta \in \mathbb{R}_{>0}$. Let $a_1 = \alpha$ and $a_{n+1} = \sqrt{\beta + a_n}$. Show that the sequence a_n converges and find the limit.

32. Let $\alpha, \beta \in \mathbb{R}_{>0}$. Let $a_1 = \alpha$ and $a_{n+1} = \beta + \sqrt{a_n}$. Show that the sequence a_n converges and find the limit.

33. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$. Show that the sequence x_n converges and find the limit.

34. Fix a real number x_1 between 0 and 1. Let $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$. Show that the sequence x_n converges and that the limit is a soluntion to the equation $x^3 - 7x + 2 = 0$. Use this observation to estimate the solution to $x^3 - 7x + 2 = 0$ to three decimal places.

35. Find the upper and lower limits of the sequence $(-1)^n\left(1 + \frac{1}{n}\right)$.

36. Find the upper and lower limits of the sequence given by $a_1 = 0$, $a_{2k} = \frac{1}{2}a_{2k+1}$, and $a_{2k+1} = \frac{1}{2} + a_{2k}$.

37. Give an example of a sequence (a_n) such that none of $\inf a_n$, $\liminf a_n$, $\limsup a_n$, and $\sup a_n$ are equal.

38. Let a_n be a bounded sequence. Show that $\liminf a_n \leq \limsup a_n$.

39. Let a_n be a bounded sequence. Show that a_n converges if and only if $\limsup a_n \leq \liminf a_n$.

40. Let a_n be a bounded sequence such that $\limsup a_n \leq \liminf a_n$. Show that $\limsup a_n = \liminf a_n = \lim a_n$.

41. Let a_n be a real sequence. Show that $\lim_{n\to\infty} a_n = a$ if and only if $\limsup a_n = \liminf a_n = a$.

42. Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

3. Convergence theorems for sequences

1. Prove that a real sequence can have at most one limit.

2. Prove that every convergent sequence is Cauchy.

3. Prove that every Cauchy sequence which has a convergent subsequence is itself convergent.

4. Prove that every Cauchy sequence is bounded.

5. Prove that every convergent sequence is bounded.

6. Prove that a contractive sequence is Cauchy.

7. Prove that a contractive sequence is convergent.

4. Series

1. Define the following and give an example for each:
   (a) series,
   (b) converges (for a series),
   (c) diverges (for a series),
   (d) limit (of a series),
   (e) absolutely convergent,
   (f) conditionally convergent,
   (g) geometric series,
   (h) harmonic series,

2. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges. Use the integral test.

3. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$ converges. Use the integral test.

4. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ converges. Use the integral test.

5. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$ converges. Use the integral test.

6. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges. Use the comparison test.

7. Determine if the series $\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$ converges. Use the comparison test.

8. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n + 1}$ converges. Use the comparison test.

9. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n - 1}$ converges. Use the comparison test.

10. Determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ converges. Use the comparison test.

11. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$ converges. Use the comparison test.

12. Determine if the series $\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$ converges. Use the comparison test.

13. Determine if the series $\sum_{n=1}^{\infty} \frac{3^n + 1}{4^n + 1}$ converges. Use the comparison test.

14. Determine if the series $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges. Use the ratio test.

15. Determine if the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. Use the ratio test.

16. Determine if the series $\sum_{n=1}^{\infty} \frac{2^n}{n + 1}$ converges. Use the ratio test.

17. Determine if the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges. Use the ratio test.

18. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ converges.

19. Determine if the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n + 1}}$ converges.

20. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^7}$ converges.

21. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + n}}$ converges.

22. Determine if the series $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ converges.

23. Determine if the series $\sum_{n=1}^{\infty} \frac{\sin n}{1 + n^2}$ converges.

24. Determine if the series $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \cdots$ converges.

25. Determine if the series $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots$ converges.

26. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n + 1)}$ converges.

27. Determine if the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$ converges.

28. Determine if the series $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ converges absolutely.

29. Determine if the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$ converges absolutely.

30. Determine if the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

31. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n + 1)}$ converges absolutely.

5. Power series

1. Write out the first four terms of the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

2. Write out the first four terms of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

3. Write out the first four terms of the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

4. Write out the first four terms of the series $\sum_{n=0}^{\infty} (\text{-}1)^n \frac{x^n}{n+2}$.

5. Find the Taylor expansion of e^x at $x = 0$.

6. Find the Taylor expansion of $\sinh x$ at $x = 0$.

7. Find the Taylor expansion of $\frac{1}{1-x}$ at $x = 0$.

8. Find the Taylor expansion of e^x at $x = 2$.

9. Find the Taylor expansion of $\log x$ at $x = 1$.

10. Find the Taylor expansion of $\frac{1}{x^2}$ at $x = 1$.

11. Prove the identity $e^{ix} = \cos x + i \sin x$.

12. Prove the identity $e^x = \cosh x + \sinh x$.

13. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

14. Find the radius of convergence of the series $\sum_{n=0}^{\infty} (\text{-}1)^n \frac{(x+1)^n}{(n+1)^2}$.

15. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

16. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n}}$.

17. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$.

18. Find the interval of convergence of the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{(n+1)^2}$.

19. Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

20. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n}}$.

21. Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

22. Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

23. Find the sum of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

24. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{3^{n-1}}$.

25. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n2^{n+1}}$.

26. Find the sum of the series $\sum_{n=1}^{\infty} n(n-1)\left(\frac{1}{4}\right)^n$.

27. Find the power series representation of $\frac{1}{1+2x}$ and determine its radius of convergence.

28. Find the power series representation of $\frac{1}{1+x^2}$ and determine its radius of convergence.

29. Find the power series representation of $\frac{x}{1+x}$ and determine its radius of convergence.

30. Find the power series representation of $\dfrac{1}{(1+x)^2}$ and determine its radius of convergence.

31. Find the power series representation of $\arctan x$ and determine its radius of convergence.

32. Find the power series representation of $\log(2+x)$ and determine its radius of convergence.

33. Find the power series representation of $\int e^{x^3}\,dx$.

34. Find the power series representation of $\displaystyle\int \frac{\sinh x}{x}\,dx$.

35. Find an infinite series representation of $\displaystyle\int_{-1}^{1} \frac{\sinh x}{x}\,dx$.

36. Find an infinite series representation of $\displaystyle\int_{0}^{1} e^{x^3}\,dx$.