University of Melbourne
Department of Mathematics and Statistics

620-295

Real Analysis with Applications

Workbook

Semester 1, 2009

Time and location of lectures

Practice classes start in the second week of the Semester. The allocation to be announced.

		Time	Venue
L	Tue	11 am	Laby Theatre
L	Wed	12 noon	Laby Theatre
L	Thu	9 am	E. Murdoch Theatre

Four Laboratory Classes will be given during the Semester on topics including convergence of sequences, iterative solution of nonlinear equations, numerical integration and Taylor polynomials and series.

Syllabus

This subject introduces the field of mathematical analysis both with a careful theoretical framework and its application in numerical approximation. A review of number systems; the fundamentals of topology of the real line; continuity and differentiability of functions of one and several variables; sequences and series including the concepts of convergence and divergence, absolute and conditional, and tests for convergence; Taylors theorem and series representation of elementary functions with application to Fourier series. The subject will introduce methods of proof such as induction and also introduce the use of rigorous numerical approximations. Topics include the definition of limits, lim sup, lim inf; Rolle's Theorem, Mean Value Theorem, Intermediate Value Theorem, monotonicity, boundedness, and the definition of the Riemann integral.

On completion of the subject the students should acquire

• an appreciation of rigour in mathematics, be able to use proof by induction, proof by contradiction, and to use epsilon-delta proofs both as a theoretical tool and a tool of approximation;

• a good knowledge of the theory and practice of power series expansions and Taylor polynomial approximations;

• an ability to numerically compute integrals based on theoretical groundwork and on practical computation using software packages

Prerequisites

One of Calculus 2, 620-143 (prior to 2009); *and one of* 620-122 (prior to 2008), 620-142 (prior to 2009), Linear Algebra, Accelerated Mathematics 1 (620-157 Mathematics 1 prior to 2009), 620-190 (UMEP Mathematics for High Achieving Students), 620-192 (prior to 2006), 620-194 (prior to 2006), 620-211 (prior to 2008)

Lecturer

Associate Professor Jerry Koliha, Room 164, Richard Berry Building
e-mail: `j.koliha@ms.unimelb.edu.au`

Tutors

Associate Professor Jerry Koliha, Room 164
Dr Penny Wightwick
Dr Allen Russell

Recommended for reference

• R. C. Wrede, M. Spiegel, *Schaum's outline of ADVANCED CALCULUS*, McGraw-Hill, US, 2002; $25.30 (as of 14/5/09)

• The subject Workbook contains a lecture-by-lecture schedule of the course, which is approximate only and may be adjusted during the semester. The Workbook also contains brief Notes on the subject.

Problem Sheets for the Practice Classes are included in the Workbook.

Assessment

Up to 50 pages of written assignments 20% (due during semester—timeline to be announced), a 3-hour written examination 80% (in the examination period).

Website

The website associated with this subject will be available from the URL

`http://www.lms.unimelb.edu.au`

Lecture-by-lecture outline

Number Systems

1. The set $\mathbb{N}$ of natural numbers. Extension to $\mathbb{Z}$. The set $\mathbb{Q}$ of rational numbers. The laws of arithmetics. Proof by contradiction (non-solvability of $x^2 = 2$ in $\mathbb{Q}$).

2. Mathematical induction. Heuristics of the domino principle. Examples. Induction for $n \geq n_0$.

3. The set $\mathbb{R}$ of real numbers—filling in gaps in $\mathbb{Q}$. Supremum and infimum. Irrational numbers, approximation by rationals. Applications.

---

4. Absolute value. Inequalities involving absolute value. AM-GM inequality for n terms. Inequalitites using calculus.

Sequences

5. Sequences; definition, examples and motivation. Heuristic limits. Limits of sequences using ε-$N(\varepsilon)$. Basic principles; algebra of limits.

6. Monotonic sequence theorems and its applications for deriving standard limits. The sandwich rule and applications.

---

7. Subsequences. Every sequence contains a monotonic subsequence. Cauchy's condition of convergence. Applications to evaluating limits.

8. Bounded sequences. Upper and lower limits. Other techniques for calculating limits of sequences: limits of sequences from limits of functions.

Functions

9. A general definition of a function. Injections, surjections, bijections. Inverse functions. Applications.

---

10. Limit of a function: heuristics and ε-$\delta(\varepsilon)$. Emphasis on approximation. Theorems on limits. Standard limits.

11. Continuity at a point. Sequential criterion of continuity. Theorems on continuity.

12. One-sided limits. Continuity on closed intervals. Uniform continuity.

---

13. Boundedness and existence of maxima and minima. Intermediate value theorem and applications. Orders of magnitude.

Differential Calculus

14. Differentiability at a point. Differentiability implies continuity. Differentiation rules. Differentiability on open intervals. Rolle's theorem.

15. Mean value theorem. l'Hôpital's rule. Applications to theorems in elementary calculus.

---

16. Orders of magnitude $o(f)$ and $O(f)$. Approximate solution to a nonlinear equation $x = f(x)$ by Picard's iterations. Newton's method for solving $F(x) = 0$.

17. Tagged partitions of intervals. Definition of the Riemann integral through Riemann sums. Riemann integrable functions: continuous, monotonic, etc. Basic properties: linearity, monotonicity. A function without the Riemann integral.

18. GOOD FRIDAY EASTER BREAK

---

19. Numerical computation of Riemann integrals: trapezoidal rule, Simpson method. Error estimates.

20. The integral as a function of the upper limit. the fundamental theorem of calculus. Mean value theorem for the integral.

21. Lebesgue's criterion of Riemann integrability. Integrals depending on parameter: continuity and differentiation. Applications.

---

22. ANZAC DAY

23. Improper integrals: infinite intervals, vertical asymptotes. Comparison test for improper integrals.

24. Improper integrals continued: Recurrence relations and improper integrals. Gamma function.

Series and Taylor polynomials

25. Infinite series. Partial sums. Convergence and divergence of series. Telescoping series. Positive term series.

26. Algebra of series. Convergence of $\sum_n a_n$ implies $a_n \to 0$, but not vice versa: harmonic series. Generalized harmonic p-series. Integral test.

27. Tests for absolute convergence of series: Comparison test, ratio test, Cauchy root test. Reordering of absolutely convergent seires.

---

28. Absolute and conditional convergence of series. Alternating series, Leibniz test. Tests for conditional convergence: Partial summation formula and Abel's test.

29. Taylor's polynomials, the remainder. Examples and applications. Approximation.

30. Approximate evaluation of integrals using Taylor's polynomials. Further applications.

Power Series

31. Power series. Radius of convergence, relation to the Cauchy root test. Differentiation and integration of a power series term-by-term. Preservation of the radius of convergence.

32. Representation of functions as power series. Taylor series. Applications to evaluation of integrals.

33. Error estimates using Taylor's expansions.

---

34. Uniform convergence of sequences and series of functions. Uniform convergence of power series.

35. Fourier series as a general method of representation of certain functions. Convergence of Fourier series. (History: Fourier, Dirichlet, Lusin and Carlsson.)

36. Miscellanaous applications and revision.

Number systems

First some notation:

$\mathbb{N}$	natural numbers (positive integers)
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{Z}$	complex numbers
$\Rightarrow$	implication
$\Leftrightarrow$	logical equivalence
$x \in A$	x is an element of A
$A \subset B$	subset: $x \in A \Rightarrow x \in B$
$A = B$	equality: $A = B \Leftrightarrow \{A \subset B \wedge B \subset A\}$
$A \cup B$	the union: $x \in A \cup B \Leftrightarrow \{x \in A \vee x \in B\}$
$A \cap B$	the intersection: $x \in A \cap B \Leftrightarrow \{x \in A \wedge x \in B\}$
$A = \bigcup_{\alpha \in \Delta} A_\alpha$	the general union: $x \in A \Leftrightarrow x \in A_\beta$ for *some* $\beta \in \Delta$
$A = \bigcap_{\alpha \in \Delta} A_\alpha$	the general intersection: $x \in A \Leftrightarrow x \in A_\alpha$ for *all* $\alpha \in \Delta$

Logical connectives $\Rightarrow$ (implies), $\Leftrightarrow$ (is equivalent), $\wedge$ (and), $\vee$ (or): Any statement f can take two values, ***T*** (true) and ***F*** (false). The connectives are described by their truth tables:

f	g	$f \Rightarrow g$	f	g	$f \Leftrightarrow g$	f	g	$f \wedge g$	f	g	$f \vee g$
T	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***
T	***F***	***F***	***T***	***F***	***F***	***T***	***F***	***F***	***T***	***F***	***T***
F	***T***	***T***	***F***	***T***	***F***	***F***	***T***	***F***	***F***	***T***	***T***
F	***F***	***T***	***F***	***F***	***T***	***F***	***F***	***F***	***F***	***F***	***F***

Note that an implication with a false premise is always true ('if $2 + 2 = 5$, then Melbourne is the capital of Australia' is true). The *negation* of a statement f is the statement $\sim f$ whose truth values are opposite of those of f. Check that $\{f \Rightarrow g\}$ is logically equivalent to $\{\sim f \vee g\}$. In mathematical proofs we often benefit by proving the implication $\{f \Rightarrow g\}$ in the equivalent form $\{\sim g \Rightarrow \sim f\}$ known as the *contraposition*.

Quantifiers. Many statements in mathematics involve the so-called logical quantifiers 'for all' ($\forall$) and 'there exists' ($\exists$). The order of these quantifiers is essential as interchanging them can alter the logical meaning of the statement. As an example we give a definition of the continuity of a function $f\colon D \to \mathbb{R}$ at a point $a \in D$. The function f is continuous at the point $a \in I$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in I)\{|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\}.$$

We often need to find the negation of this statement. It can be shown that this is equivalent to interchanging $\forall$ and $\exists$:

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in I)\{|x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon\}.$$

We had to take the negation of $\{u \Rightarrow v\}$, which is $\{u \wedge \sim v\}$ (check). This procedure needs a bit of practice, but is very useful in certain proofs.

Laws of arithmetics If a set S is equipped with two operations $+$ (addition) and $\cdot$ (multiplication), we can consider the following laws:

$(a+b)+c = a+(b+c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	associative laws
$x+y = y+x$, $x \cdot y = y \cdot x$	commutative laws
$(a+b) \cdot c = a \cdot c + b \cdot c$, $c \cdot (a+b) = c \cdot a + c \cdot b$	distributive laws

Two distributive laws are needed if the commutative law for multiplication is not valid (for instance multiplication of square matrices), otherwise only one will do. An element $u \in S$ is called a *unit* if $a \cdot u = a = u \cdot a$ for all $a \in S$, and an element $z \in S$ is called a *zero* if $a + z = a = z + a$ for all $a \in S$. We shall write 1 for the unit and 0 for the zero (if they are unique).

Given an element $a \in S$, $a' \in S$ is an *opposite* of a if $a + a' = 0 = a' + a$, and $a^* \in S$ is called an *inverse* or a *reciprocal* of a if $a \cdot a^* = 1 = a^* \cdot a$. We observe that $(a')' = a$ and $(a^*)^* = a$.

Natural numbers. The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of *natural numbers* obeys both commutative laws, both associative laws, and the distributive law (one is enough); $\mathbb{N}$ does not have a zero, but it has the unit 1. Let us define a relation between any two elements $a, b \in \mathbb{N}$:

$$a < b \overset{\text{def}}{\Longleftrightarrow} b = a + x \text{ for some } x \in \mathbb{N}.$$

This relation defines an *order* on $\mathbb{N}$ with the following properties:

1. For any $a, b \in \mathbb{N}$ one and only one of the following holds:

$$a < b, \qquad a = b, \qquad b < a.$$

2. $\{a < b \wedge b < c\} \Rightarrow a < c$ (transitive law)
3. $a < b \Rightarrow \{a + c < b + c\}$ for any $c \in \mathbb{N}$
4. $a < b \Rightarrow \{a \cdot c < b \cdot c\}$ for any $c \in \mathbb{N}$

We shall write $a > b$ for $b < a$. Given natural numbers a, b, the equation

$$a + x = b \tag{1}$$

is solvable for some $x \in \mathbb{N}$ if and only if $b > a$.

Integers. In order to be able to solve equation (1) for any pair a, b, we extend the natural numbers to include the zero 0, and for each element $a \in \mathbb{N}$ include its opposite a'. The elements of the extended set $\mathbb{Z}$ are called *integers*:

$$0, 1, 1', 2, 2', 3, 3', \ldots$$

The commutative and associative laws are retained in $\mathbb{Z}$, as is the distributive law. We can now solve the equation (1) without any restriction: Given $a, b \in \mathbb{Z}$, set $x = a' + b$. Then

$$a + x = a + (a' + b) = (a + a') + b = 0 + b = b.$$

We introduce a new operation of *subtraction* by

$$a - b := a' + b = b + a';$$

it is then convenient to write $-a$ instead of a'. The minus sign then has two roles: It denotes the opposites as well as the operation of subtraction. It is easily checked that $b - (-a) = b - a' = b + (a')' = b + a$.

We can introduce an order on $\mathbb{Z}$ by defining

$$a < b \overset{\text{def}}{\Longleftrightarrow} b - a \in \mathbb{N}.$$

It has the properties 1–3 listed for the set $\mathbb{N}$ earlier. Property 4 does not hold in general, but has to be modified as follows:

$$(a < b) \wedge (c > 0) \Rightarrow a \cdot c < b \cdot c.$$

Next we consider the equation

$$a \cdot x = b, \qquad a \neq 0, \tag{2}$$

for a given pair $a, b \in \mathbb{Z}$, $a \neq 0$. Such equation is not always solvable by an element $x \in \mathbb{Z}$. If $a = 2$ and $b = 7$, then there is no $x \in \mathbb{Z}$ satisfying $2 \cdot x = 7$. It is customary to omit the dot for multiplication and write simply ab for $a \cdot b$.

Rational numbers. To be able to solve equation (2) for any pair a, b (provided $a \neq 0$), we extend the integers to rational numbers. We briefly describe a construction of rationals from integers using ordered pairs of integers. Let $\mathbb{Q}$ be the set of all ordered pairs (a, b), where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. We define equality, addition and multiplication using only operations in $\mathbb{Z}$ as follows:

$$\begin{aligned} (a, b) = (c, d) &\overset{\text{def}}{\Longleftrightarrow} ad = bc, \\ (a, b) + (c, d) &= (ad + cb, bd), \\ (a, b)(c, d) &= (ac, bd). \end{aligned}$$

It can be verified that $\mathbb{Q}$ obeys the commutative, associative and distributive laws, with the zero $(0, 1)$ and unit $(1, 1)$; each element (a, b) has its opposite $(-a, b)$, and each nonzero element (a, b) has its inverse (b, a). The equation (2) is always solvable. For this we rewrite it as $(a, b)(x, y) = (c, d)$, where all pairs belong to $\mathbb{Q}$ and $c \neq 0$. The solution is then $(x, y) = (b, a)(c, d)$. The usual notation for rational numbers is

$$(a, b) = \frac{a}{b}.$$

An order in $\mathbb{Q}$ is introduced by

$$\frac{a}{b} < \frac{c}{d} \Leftrightarrow \frac{ab}{b^2} < \frac{cd}{d^2} \overset{\text{def}}{\Longleftrightarrow} abd^2 < cdb^2$$

where the operations on the right are performed in $\mathbb{Z}$. (The middle step is to ensure $b^2 > 0$ and $d^2 > 0$ in $\mathbb{Z}$.) It can be checked that the order has the requisite properties.

It is convenient to introduce the following notation:

$$a \leq b \overset{\text{def}}{\Longleftrightarrow} a < b \vee a = b.$$

We recall that every rational number can be expressed in the decimal system by means of a decimal expansion; such expansion is either finite or periodic.

Example 1. The rational number 5/8 has a decimal expansion

$$\frac{5}{8} = 0.625,$$

while

$$\frac{34241}{99900} = 0.34275275275\cdots = 0.34\overline{275},$$

where the overline signifies periodic repetition of the digits 275 in the expansion. Conversely, every finite or periodic expansion is a rational number. For instance,

$$0.23\overline{5} = \frac{235 - 23}{900} = \frac{212}{900} = \frac{53}{225}.$$

Real numbers. We consider the equation

$$x^2 = a, \qquad a > 0. \tag{3}$$

If $a \in \mathbb{Q}$, we cannot guarantee that there is a solution $x \in \mathbb{Q}$. To see this we prove the following result by a method known as the *proof by contradiction*: To prove implication $f \Rightarrow g$ it is enough to ensure that the combination $\{f \text{ is true}\} \wedge \{g \text{ is false}\}$ does not occur. So we assume that f is true and g false, and try to deduce a false statement (called a contradiction).

Theorem 1. *There is no rational number x satisfying $x^2 = 2$.*

Proof. For a proof by contradiction assume that such $x = a/b$ exists. We assume that both a and b are natural numbers and that a, b have no common factor other than 1 (this can be always achieved in view of the definition of equality of rationals). Then $a^2 = 2b^2$, that is, a^2 is even. But a itself cannot be odd, since in this case we would have $a^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1$, and a^2 would be odd. So a is even, $a = 2n$ for some $n \in \mathbb{N}$. From $a^2 = 2b^2$ we conclude that $4n^2 = 2b^2$ and $b^2 = 2n^2$. So b^2 is even, and so b itself is even. But now a and b have a common factor 2, which we carefully excluded at the beginning. This contradiction shows that our original assumption is false, that is, there is no rational number x satisfying $x^2 = 2$.

When we plot all rational numbers as points on a line, there will be many gaps. One such gap is due to the fact that there is no rational number whose square is equal to 2. We can divide all rational numbers to two sets:

$$A = \{x \in \mathbb{Q} : (x \leq 0) \vee [(x > 0) \wedge (x^2 < 2)]\},\ B = \{x \in \mathbb{Q} : (x > 0) \wedge (x^2 > 2)\}. \quad (4)$$

If the positive solution x to the equation $x^2 = 2$ existed in $\mathbb{Q}$, it would be greater than any element of A and less than any element of B. But such a solution can be constructed using infinite non-periodic expansion:

$$\begin{aligned} 1.4^2 = 1.96 < 2, &\quad 1.5^2 = 2.25 > 2 \\ 1.41^2 = 1.9881 < 2, &\quad 1.42^2 = 2.0164 > 2 \\ 1.414^2 = 1.999396 < 2, &\quad 1.415^2 = 2.002225 > 2 \end{aligned}$$

Continuing this way we get $1.414213562\ldots$ with an option to proceed indefinitely, and thus obtain (given infinite time) every decimal in the expansion of a new number x, which will represent a solution to the equation $x^2 = 2$. Observe that what we ever get is a rational approximation to this solution, more precise with every new decimal. This is an example of an *irrational number*, which is impossible to get in its entirety, only to be approximated by rational numbers, theoretically with an arbitrary precision. The usual notation for this number is $x = \sqrt{2}$.

We can now describe *real numbers* by infinite decimal expansions, rational numbers having periodic expansions, irrational numbers non-periodic expansions. Irrational numbers come in two varieties: *algebraic irrational numbers*, which are solutions to equations of the form $p(x) = 0$, where p is a polynomial with rational coefficients, and *transcendental irrational numbers*, which are not zeros to any polynomial with rational coefficients. Examples of transcendental numbers are π and Euler's number e.

We now discuss the effect of filling in the gaps in $\mathbb{Q}$ and compare the sizes of $\mathbb{Q}$ and $\mathbb{R}$. We say that two sets A and B are *equipotent* (have the same size) if there is a one-to-one correspondence $f\colon A \to B$. Any set equipotent to the set $\mathbb{N}$ of natural numbers is said to be *countably infinite*. Such sets are infinite, but relatively small. Perhaps surprisingly, $\mathbb{Q}$ is countably infinite, given that it is so dense in the real line. First we can consider the set $\mathbb{Q}^+$ of all positive rational numbers. In the first row we put all positive rational with the denominator 1, in the second row all positive rationals in the reduced form with the denominator 2 with the exception of those which already appeared in the first row, in the third row we place all positive rationals in the reduced form with the denominator 3 with the exceptions of those which already appeared in row 1 or row 2, etc.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \ldots \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \ldots \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & \frac{7}{3} & \ldots \\ \frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & \frac{9}{4} & \ldots \end{array}$$

We can now place the positive rationals in a one-to-one cerrespondence with the natural numbers in many ways. For example,

$$1 \to 2 \to \tfrac{3}{2} \to \tfrac{1}{2} \to \tfrac{1}{3} \to \tfrac{2}{3} \to \tfrac{4}{3} \to \tfrac{5}{2} \to 3 \to 4 \to \tfrac{7}{2} \to \tfrac{5}{3} \to \tfrac{7}{4} \to \tfrac{5}{4} \to \tfrac{3}{4} \to \tfrac{1}{4} \to \cdots$$

Now it is not difficult to place all rationals in a one-to-one correspondence with $\mathbb{N}$, thus showing that $\mathbb{Q}$ is countably infinite. But the same cannot be done with $\mathbb{R}$. We can show using a contradiction that even just real numbers between 0 and 1 cannot be put in a one-to-one correspondence with $\mathbb{N}$, and that $\mathbb{R}$ is of much larger size than $\mathbb{Q}$. For this we follow a clever procedure known as Cantor's diagonal argument, due to the great German mathematician Georg Cantor (1845–1918).

For a proof by contradiction we assume that all real numbers between 0 and 1 can be put in a one-to-one correspondence with $\mathbb{N}$, and list them in their infinite decimal expansions:

$$0.a_{11}a_{12}a_{13}\dots$$
$$0.a_{21}a_{22}a_{23}\dots$$
$$0.a_{31}a_{32}a_{33}\dots$$

To ensure uniqueness, we exclude expansions with periodic 0, that is, 3.10000... will be written as 3.099999... Now we take the number defined by the decimal expansion

$$0.\tilde{a}_{11}\tilde{a}_{22}\tilde{a}_{33}\dots$$

where $\tilde{a}_{kk} \neq a_{kk}$ for all k, taking care that the expansion is in an admissible form. This number is between 0 and 1, but is not in our list, which is supposed to contain *all* real numbers between 0 and 1. The contradiction shows that the above ordering is impossible. In fact, $\mathbb{R}$ is enormously larger than $\mathbb{Q}$.

Decimal expansions are not very convenient from the point of view of algebra, and so we give a description in terms of algebraic laws.

Algebraic description of the field $\mathbb{R}$ of real numbers. The real numbers form a set $\mathbb{R}$ with two special elements 0 and 1 and two operations, addition $x + y$ and multiplication xy, satisfying the following laws:

- A1 $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$ (associative laws)
- A2 $a + b = b + a$, $ab = ba$ for all $a, b \in \mathbb{R}$ (commutative laws)
- A3 $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{R}$ (the law of zero and unit)
- A4 for any $a \in \mathbb{R}$ there is $a' \in \mathbb{R}$ such that $a + a' = 0$ (the law of opposite);
  for any $a \in \mathbb{R} \setminus \{0\}$ there is $a^* \in \mathbb{R}$ such that $aa^* = 1$ (the law of inverse)
- A5 $a(b + c) = ac + bc$ for any $a, b, c \in \mathbb{R}$ (distributive law)

As usual, the opposite a' will be denoted by $-a$, and the inverse (reciprocal) a^* by a^{-1} or $1/a$.

The set $\mathbb{R}$ contains a subset $\mathbb{R}^+$, called the set of *positive real numbers* with two properties:

O1 Given any $a \in \mathbb{R}$, exactly one of the following is true:
$a \in \mathbb{R}^+, \quad a = 0, \quad -a \in \mathbb{R}^+.$

O2 If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$.

According to O1, $0 \notin \mathbb{R}^+$. What about 1? By O1, either $1 \in \mathbb{R}^+$ or $-1 \in \mathbb{R}^+$, but not both. Suppose that $-1 \in \mathbb{R}^+$. By O2, $1 = (-1)\cdot(-1) \in \mathbb{R}^+$, which is impossible. This contradiction shows that $-1 \in \mathbb{R}^+$ is false. Thus $1 \in \mathbb{R}^+$.

We define an *order relation* on $\mathbb{R}$ as follows:

$$a < b \overset{\text{def}}{\iff} b - a \in \mathbb{R}^+.$$

It is convenient to write $b > a$ for $a < b$. Then $a \in \mathbb{R}^+$ is equivalent to $a > 0$.

It is not difficult to verify the following properties of the order relation.

(i) Given $a, b \in \mathbb{R}$, exactly one of the following is true:

$$a < b, \quad a = b, \quad b < a.$$

(ii) $\{a < b \wedge b < c\} \Rightarrow a < c$ (transitive law)

(iii) $a < b \Rightarrow a + c < b + c$ (additive monotonicity)

(iv) $\{a < b \wedge c > 0\} \Rightarrow ac < bc$ (multiplicative monotonicity)

From (iv) it follows that $\{a < b \wedge c < 0\} \Rightarrow ac > bc$ (reversal of order). As before, $a \leq b$ means $a < b$ or $a = b$.

The set $\mathbb{R}$ contains the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. Indeed, $\mathbb{N}$ is represented in $\mathbb{R}$ as the unit 1, and then 2:=1+1, 3:= 1+1+1, ...; the other two sets are constructed from $\mathbb{N}$ as explained earlier.

Let S be a subset of $\mathbb{R}$. A real number u is an *upper bound for S* if $x \leq a$ for all $x \in S$. A set S is *bounded above* if it has an upper bound. Similarly, $l \in \mathbb{R}$ is a *lower bound* for S if $l \leq x$ for all $x \in S$. A set S is *bounded below* if it has a lower bound.

Further, $M \in \mathbb{R}$ is a *supremum* (*least upper bound*) for S if

(i) M is an upper bound of S, and

(ii) $M \leq u$ for any upper bound u of S.

Similarly, m is an *infimum* (*greatest lower bound*) for S if

(i) m is a lower bound of S, and

(ii) $l \leq m$ for any lower bound l of S.

If S has a supremum, it must be unique; the same goes for infimum.

Example 2. If S is the interval $[0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$, then the supremum of S is equal to 1. Observe that 1 does *not* belong to S. The infimum of S is 0, but it does belong to S. We have a special terminology for this: The supremum which belongs to the set S is called the *maximum* of S, and the infimum which belongs to the set S is the *minimum* of S. In our case, $S = [0, 1)$ has the minimum 0, but has no maximum.

The following order axiom makes sure that there are no gaps in the set $\mathbb{R}$.

Order completeness axiom for $\mathbb{R}$. Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

This axiom does not hold in the set $\mathbb{Q}$ of rational numbers. If we take the subset A of $\mathbb{Q}$ defined in (4), we see that A is nonempty ($1 \in A$), and bounded above, for example by 4. Yet it does not have a supremum in $\mathbb{Q}$, for such a supremum $x \in \mathbb{Q}$ would satisfy $x^2 = 2$.

Theorem 2. *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

Proof. Suppose that $\mathbb{N}$ has an upper bound. By the order completeness axiom, there exists $M = \sup \mathbb{N}$. Then $M - 1 < M$ and hence $M - 1$ is not an upper bound of $\mathbb{N}$, that is, there is $n \in \mathbb{N}$ such that $M - 1 < n$. But then $n + 1 \in \mathbb{N}$ and $M < n + 1$, which is a contradiction since M is an upper bound of $\mathbb{N}$.

Inequalities Inequalities are the foundation on which mathematical analysis is built. A good grasp of inequalities is essential for understanding the subject. The key role is played by *absolute value* of a real number or more generally of a complex number. Given $a \in \mathbb{R}$, we define

$$|a| := \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Another way of looking at the absolute value is to use the concept of maximum:

$$|a| = \max\{a, -a\}.$$

The maximum of a set $S \subset \mathbb{R}$ (if it exists) is the greatest element of S. In other words, the maximum of S is the supremum of S which *belongs to* S. The so-called *triangle inequality* states

$$|a + b| \leq |a| + |b|.$$

To prove this we note that $a \leq |a|$ and $b \leq |b|$. Hence $a + b \leq |a| + |b|$. Similarly, $-a \leq |a|$ and $-b \leq |b|$, so that $-(a + b) \leq |a| + |b|$. Consequently,

$$|a + b| = \max\{a + b, -(a + b)\} \leq |a| + |b|.$$

To solve inequalities involving the absolute value, first we have to transcribe them removing the absolute value. Te key inequality here is

$$|x| < a \Leftrightarrow -a < x < a,$$

where $a > 0$.

Example 3. Find all $x \in \mathbb{R}$ satisfying $|x+2| < 5$. First the removal of the absolute value:

$$-5 < x + 2 < 5;$$

these are two inequalities holding simultaneously: $-5 < x + 2$ and $x + 2 < 5$. Adding -2 to all sides of the inequality, we get $-7 < x < 3$. This may be rewritten as $x \in (-7, 3)$.We can often use calculus to prove an inequality, especially when it involves functions.

Example 4. Show that $\log x \leq x - 1$ for all $x > 0$. The next step involves a very useful principle: We set

$$f(x) = (x - 1) - \log x, \quad x > 0,$$

and differentiate $f(x)$ to find its raising and falling:

$$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x};$$

thus $f'(x) \geq 0$ if $x \geq 1$ and $f'(x) \leq 0$ if $0 < x \leq 1$. This means that f is decreasing on $(0, 1]$ and increasing on $[1, \infty)$, and so $f(x) \geq f(1) = 0$ for all $x > 0$. This proves the required inequality.

Example 5. Prove that

$$\frac{1}{n+1} + \log n \leq \log(n+1) \quad \text{for all } n \in \mathbb{N}.$$

We can try to use induction, but in this case it is not helpful (try it!). Rewrite the inequality as

$$\frac{1}{n+1} \leq \log(n+1) - \log n = \log \frac{n+1}{n},$$

$$\frac{\frac{1}{n}}{1 + \frac{1}{n}} \leq \log\left(1 + \tfrac{1}{n}\right),$$

$$\frac{x}{1+x} \leq \log(1+x)$$

with $x = 1/n$. We pass from the discrete variable $1/n$ to the continuous variable x in order to use calculus. We set

$$f(x) = \log(1+x) - \frac{x}{1+x}, \quad x \geq 0$$

and differentiate to find the raising and falling of f:

$$f'(x) = \frac{1}{1+x} - \frac{1+x-1}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} \geq 0$$

if $x \geq 0$. This show that f is increasing for $x \geq 0$; since $f(0) = 0$, we have $f(x) \geq f(0) = 0$ for all $x \geq 0$. The inequality is proved.

Limits of sequences. (Cauchy.) A sequence (a_n) of real number has a *limit* $a \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n > N(\varepsilon).$$

A sequence which has a limit is called *convergent*, a sequence with no limit is *divergent*. We write

$$a = \lim_{n\to\infty} a_n \quad \text{or} \quad a_n \to a.$$

Limit theorems. *Suppose that $a_n \to a$, $b_n \to b$ and $c \in \mathbb{R}$. Then*

(i) $a_n + b_n \to a + b$, $a_n b_n \to ab$ and $ca_n \to ca$

(ii) $1/a_n \to 1/a$ if $a \neq 0$ ($1/a_n$ may not be defined for a finite number of n)

(iii) $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n$

Let us prove (ii) as a sample. Let $\varepsilon > 0$ be given. By the definition of limit, there is N_1 such that $|a_n - a| < \varepsilon$ if $n > N_1$. Further, there is N_2 such that $|a_n - a| < \frac{1}{2}|a|$ whenever $n > N_2$. Let $N_0 = \max\{N_1, N_2\}$. Then for any $n > N_0$, $|a_n| = |a + (a_n - a)| \geq |a| - |a_n - a| > |a| - \frac{1}{2}|a| = \frac{1}{2}|a|$, so that $1/|a_n| < 2/|a|$, and

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n|\,|a|} < \frac{2}{|a|^2}\,\varepsilon \quad \text{if } n > N_0.$$

Since $2/|a|^2$ is a constant, this proves the result.

Theorem 3 (Sandwich theorem)**.** *Suppose that $a_n \to a$, $b_n \to a$ and $a_n \leq c_n \leq b_n$ for all n, Then (c_n) converges and $c_n \to a$.*

Proof. For all n, $a_n - a \leq c_n - a \leq b_n - a$. By the definition of limit there exists $N(\varepsilon)$ such that $|a_n - a| < \varepsilon$ and $|b_n - a| < \varepsilon$ if $n > N(\varepsilon)$. Then $-\varepsilon < c_n - a < \varepsilon$ for all $n > N(\varepsilon)$.

Infinite limits. A sequence (a_n) is said to *diverge to* $+\infty$ if $a_n > 0$ from a certain index on, and $\lim_{n\to\infty}(1/a_n) = 0$; it is said to *diverge to* $-\infty$ if $a_n < 0$ from a certain index on, and $\lim_{n\to\infty}(1/a_n) = 0$. For instance,

$$\lim_{n\to\infty} n^2 = +\infty, \qquad \lim_{n\to\infty}(12n - n^3) = -\infty;$$

the sequence $(-1)^n n^4$ oscillates and has no limit, finite or infinite.

A real sequence (a_n) is *increasing* (*strictly increasing*) in $a_n \leq a_{n+1}$ for all n ($a_n < a_{n+1}$ for all n). *Decreasing* and *strictly decreasing* sequences are defined analogously. A sequence which is either increasing or decreasing is called a *monotonic sequence.*

Theorem 4 (Monotonic sequence theorem)**.** *A monotonic bounded sequence is convergent.*

Proof. Suppose that (a_n) is increasing and bounded above. By the order completeness of $\mathbb{R}$, there exists $a = \sup\{a_n : n \in \mathbb{N}\}$. By the characterization of the supremum (Problem 3), for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $a \geq a_N > a - \varepsilon$. For all $n > N$ we have $a_n \geq a_N$ (increasing sequence), and so $a - \varepsilon < a_N \leq a_n \leq a < a + \varepsilon$. Hence $|a_n - a| < \varepsilon$ for all $n > N$. If (a_n) is decreasing, apply the preceding result to the increasing sequence $(-a_n)$.

Theorem 5. *Every real sequence contains a monotonic subsequence.*

Proof given in the Lectures.

A sequence is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_m - a_n| < \varepsilon \quad \text{for all } m, n > N(\varepsilon).$$

Theorem 6 (Bolzano–Weierstrass theorem)**.** *Every bounded seqence contains a convergent subsequence.*

Proof given in the Lectures.

Theorem 7 (Bolzano–Cauchy theorem)**.** *Every Cauchy sequence is convergent.*

Proof given in the Lectures.

The Bolzano–Cauchy theorem enables us to decide whether a sequence is convergent even when we do not know the value of the limit. This is often very useful.

Example 6. Define a sequence (a_n) by

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{a_n}, \quad n = 1, 2, \ldots$$

We show that (a_n) is a Cauchy sequence. After some calculations, we can show that

$$|a_{n+1} - a_n| = \frac{|a_{n-1} - a_n|}{|a_n a_{n-1}|} \leq \tfrac{1}{2}|a_n - a_{n-1}|, \quad n \geq 2.$$

If $m > n$, then

$$\begin{aligned} |a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\ &\leq |a_2 - a_1| \left((\tfrac{1}{2})^{m-2} + (\tfrac{1}{2})^{m-3} + \cdots + (\tfrac{1}{2})^{n-1}\right) \\ &\leq |a_2 - a_1|(\tfrac{1}{2})^{n-2}; \end{aligned}$$

from this we can deduce that (a_n) is Cauchy. By the Bolzano–Cauchy theorem the sequence converges. With this knowledge we can calculate the limit explicitly: Write $a = \lim_{n\to\infty} a_n$. Then, using limit theorems, we have

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(1 + \frac{1}{a_n}\right) = 1 + \frac{1}{a};$$

this gives $a = 1 + 1/a$, and $a = (1 + \sqrt{5})/2$.

Let (a_n) be a bounded sequence. We define the *upper limit* or *limit superior* $\limsup_{n\to\infty} a_n$ as follows: Form the sequence $b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ for $n \in \mathbb{N}$, and observe that the sequence (b_n) is decreasing (suprema on successively smaller sets) and bounded below. Thus (b_n) is convergent by the monotonic sequence theorem, and we set

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} b_n.$$

Analogously we define the *lower limit* or the *limit inferior* of a bounded sequence (a_n). Succinctly,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k, \qquad \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

Functions. Let A, B be two sets. A function $f\colon A \to B$ is a rule which assigns to each element $x \in A$ a unique element $f(x)$ in B. The set A is the *domain* of f and B is the *codomain* of f. For two functions to be *equal*, they have to agree in the domain, codomain and the rule. A function $f\colon A \to B$ is *injective* if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B; f is *surjective* if for each $y \in B$ there exists $x \in A$ such that $f(x) = y$. A function $f\colon A \to B$ is *bijective* if it is both injective and surjective. Changing the domain or codomain we can change the injectivity or surjectivity of the function.

Example 7. Let $f\colon \mathbb{R} \to \mathbb{R}$ have the rule $f(x) = x^2$. Then f is neither injective ($f(-x) = (-x)^2 = x^2 = f(x)$ for any $x \in \mathbb{R}$) nor surjective (for $y = -5$ in $\mathbb{R}$ there is no $x \in \mathbb{R}$ such that $f(x) = -5$). But keep the rule, and change the domain or codomain: $g(x) = h(x) = k(x) = x^2$, and

$$g\colon [0,\infty) \to \mathbb{R}, \qquad h\colon \mathbb{R} \to [0,\infty), \qquad k\colon [0,\infty) \to [0,\infty).$$

Then g is injective but not surjective, h is surjective but not injective, and k is bijective. (Sketch the graphs for all the functions.) This tells us that f, g, h, k are four genuinely different functions.

If $f\colon A \to B$ is bijective, there exists a unique function $g\colon B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$; g is called the *inverse* function to f.

With the correct understanding of the concept of the function, we use a certain flexibility in notation, as the rigid adhesion to the rigorous definition would cause a notation clutter. Most functions considered in this course are defined on a real interval I with the codomain a subset of $\mathbb{R}$ (including $\mathbb{R}$ itself). We shall usually write $f\colon I \to \mathbb{R}$; if f is injective, we may use the same letter f to denote the *range restriction* of f, $f\colon I \to f(I)$, which will make the function bijective for the purpose of constructing its inverse.

Limits of functions. (Cauchy's definition.) Let $f\colon I \to \mathbb{R}$ be a real valued function defined on an interval I, and let a be an interior point of I. We say that $L \in \mathbb{R}$ is the *limit* of f at a if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

It is essential that the point a itself be *excluded* from the consideration, as the value of $f(a)$ is not relevant to the definition of the limit. The function f may not be even defined at a. We write $L = \lim_{x\to a} f(x)$ or $f(x) \to L$ as $x \to a$.

(Heine's definition.) Equivalent definition due to Heine is given in terms of sequences: $f(x) \to L$ as $x \to a$ if and only if

$$\{x_n \in I \setminus \{a\},\ x_n \to a\} \Rightarrow f(x_n) \to L.$$

This definition is often useful since we can apply known theorems on sequences.

Example 8. Using Cauchy's definition of the limit show that

$$\lim_{x\to 2} \frac{x^2+1}{2x-1} = \frac{5}{3}.$$

Let $\varepsilon > 0$ be given. Suppose that $0 < |x-2| < \delta$ for some, as yet unknown δ, to be determined in dependence on ε. It is often convenient to assume that δ is small, say $0 < \delta < 1$; this does not affect the definition of the limit, and we shall so assume. It is also convenient to write $x = 2 + h$, so that $0 < |h| < \delta$. We have

$$\left|\frac{x^2+1}{2x-1} - \frac{5}{3}\right| = \left|\frac{5+h(h+4)}{3+2h} - \frac{5}{3}\right| = \frac{|h|\,|3h+2|}{3|3+2h|} < 5\delta,$$

as $|3h+2| \le 3|h|+2 < 5$ and $|3+2h| \ge 3-2|h| > 3-2 = 1$. If we choose $\delta = \min\{\frac{3}{5}\varepsilon, 1\}$, we get $0 < |x-2| < \delta \Rightarrow |f(x) - \frac{5}{3}| < \varepsilon$.

There are limit theorems for functions analogous to those for sequences (see Problem 48).

Continuity of functions. A functions f is continuous at a point a, an interior point of the domain of f, if the limit of f at a exists and equals the function value:

$$\lim_{x\to a} f(x) = f(a).$$

this can be expressed in terms of sequences as

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

Observe carefully that the limit and function can be interchanged if and only if the function is continuous. Theorems on continuity parallel those on limits (see Problem 48).

Theorem 8 (Intermediate value theorem)**.** *Let $f\colon I \to \mathbb{R}$ be a function continuous on an interval I. If $a, b \in I$, $a \neq b$, and w lies between $f(a)$ and $f(b)$, then there exists $x \in I$ such that $f(x) = w$.*

Proof. Without a loss of generality we may assume that $a < b$ and $f(a) < f(b)$. If $f(a) < w < f(b)$, define

$$S := \{x \in [a,b] : f(x) < w\}.$$

The set S is nonempty and bounded above by w, therefore is possesses supremum $c = \sup S$. By Problem 3 there is a sequence (x_n) in S such that $x_n \to c$. By continuity, $f(x_n) \to f(c)$. Since $f(x_n) < w$ for all n, we have $f(c) \le w$ (in the limit a strict inequality may become equality). Define $b_n = c+(b-c)/n$. Then $b_n > c$ and $b_n \to c$, so that $f(b_n) \to f(c)$ where $f(b_n) > w$ for all n. Hence $f(c) \ge w$. Together with the previous inequality this implies $f(c) = w$.

Uniform continuity. Let $f\colon I \to \mathbb{R}$ be continuous at each point of the interval I.

Then for each $\varepsilon > 0$ and each point $a \in I$ there exists $\delta(a)$ (dependent on ε and a) such that

$$|x - a| < \delta(a) \Rightarrow |f(x) - f(a)| < \varepsilon.$$

If $f\colon (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/x$, then for a given $\varepsilon > 0$ the corresponding $\delta(a)$ must be chosen smaller and smaller as the point a approaches 0. (Sketch.) However, if for a given $\varepsilon > 0$ there is one δ which works for all points in I, we say that f is *uniformly continuous on I*. This means that for each $\varepsilon > 0$ there exists $\delta > 0$ dependent only on ε such that

$$x, y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Theorem 9. *Let $I = [a, b]$ be a closed and bounded interval, and let $f\colon I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Proof. Write the statement of the uniform continuity in terms of quantifiers:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in I)(\forall y \in I) : \{|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\}. \tag{5}$$

For a proof by contradiction we assume that f is not uniformly continuous. So the negation of the above statement is true:

$$(\exists\varepsilon > 0)(\forall\delta > 0)(\exists x \in I)(\exists y \in I) : \{|x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon\}. \tag{6}$$

Here ε is a fixed quantity. Choose δ in succession to be $1, \frac{1}{2}, \dots, \frac{1}{n}, \dots$ Then for each n there exist points x_n, y_n in I such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. By the Bolzano–Weierstrass theorem for closed bounded intervals we can extract a convergent subsequence from (x_n), and then from the corresponding subsequence of (y_n) again a convergent subsequence. Thus we have subsequences (x_{k_n}) and (y_{k_n}), both convergent to the same element $z \in I$ as $|x_{k_n} - y_{k_n}| < 1/k_n \to 0$. Hence by the continuity of f, $f(x_{k_n}) \to f(z)$ and $f(y_{k_n}) \to f(z)$. But this is impossible since $|f(x_{k_n}) - f(y_{k_n})| \geq \varepsilon$ for all n. This contradiction shows that non-uniform continuity of f is not possible, and the result follows.

Note. If we interchange the second and third quantifier in (5), we obtain the *ordinary* (not uniform) continuity on I as $\delta > 0$ depends on both ε and x.

Continuity on closed bounded sets. A set $A \subset \mathbb{R}$ is *closed* if it has the following property:

$$[(x_n) \text{ any sequence in } A \text{ and } x_n \to x] \Rightarrow x \in A.$$

Examples of closed sets are intervals of the form $[a, b]$, $[a, \infty)$, $(-\infty, b]$. A more exotic example is the set $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

Theorem 10. *Let D be a closed bounded subset of $\mathbb{R}$ and let $f\colon D \to \mathbb{R}$ be continuous on D. Then:*

(i) *f is a bounded function.*

(ii) *f attains its maximum and minimum on D.*

(iii) *f is uniformly continuous on D.*

Proof. (i) For a proof by contradiction assume that f is not bounded. Then there is a sequence (x_n) in D such that $|f(x_n)| > n$. The sequence (x_n) is bounded since D is a bounded set. By the Bolzano–Weierstrass theorem, (x_n) has a convergent subsequence (x_{k_n}). If $x_{k_n} \to x$, then $x \in D$ since D is closed. By the continuity of f on D, $f(x_{k_n}) \to f(x)$. But this is impossible since $|f(x_{k_n})| > k_n$, and $k_n \to \infty$.
(ii) Since f is bounded on D, there are m and M such that

$$m = \inf\{f(x) : x \in D\}, \qquad M = \sup\{f(x) : x \in D\}.$$

By Problem 3 there are sequences (u_n) and (v_n) in D such that $f(u_n) \to m$ and $f(v_n) \to M$. Applying the Bolzano–Weierstrass theorem and the closedness of D again, we conclude that there are points u, v in D such that $f(u) = m$ and $f(v) = M$. (Supply details.)

(iii) Examining the proof of theorem on the uniform continuity on $[a, b]$, we observe that it applies also to a closed bounded set D in place of $[a, b]$.

Orders of magnitude. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, it is of interest to compare the speed with which the two functions approach 0. For example both x^2 and x^3 converge to 0 as $x \to 0$, but x^3 converges faster to 0 than x^2. More generally we compare the behaviour of two functions near a given point a using the concept of the 'order of magnitude'. We use the expression 'near a' to mean 'in some open interval containing a'. Suppose f, g are real valued functions defined near a.

$$f(x) = O(g(x)) \text{ as } x \to a \overset{\text{def}}{\Leftrightarrow} |f(x)| \le K|g(x)| \text{ near } a \text{ for some } K > 0,$$

$$f(x) = o(g(x)) \text{ as } x \to a \overset{\text{def}}{\Leftrightarrow} \lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

If $f = O(g)$ as $x \to a$, we say that the order of magnitude of f does not exceed the order of magnitude of g near a; if $f = o(g)$ as $x \to a$, we say that the order of magnitude of f is less than the order of magnitude of g near a.

For example, $\log x = o(x)$ as $x \to \infty$, and $\sin x = O(x)$ as $x \to 0$. We use the symbol $O(1)$ as $x \to a$ for any function bounded near a, and $o(1)$ as $x \to a$ for any function convergent to 0 as $x \to a$. If $f = O(g)$ and $g = O(f)$ as $x \to a$, we say that f and g are *asymptotic* at a, written $f \asymp g$ as $x \to a$.

Differentiability. A function f is *differentiable* at a if there is a constant L such that

$$f(x) = f(a) + L(x - a) + o(x - a) \text{ as } x \to a.$$

This means that near a the function f is closely approximated by the line $y = f(a) + L(x - a)$.

To obtain the constant L we write

$$L = \frac{f(x) - f(a)}{x - a} - \frac{o(x - 1)}{x - a}.$$

From the definition of $o(x-a)$ we have $\lim_{x\to a} o(x-a)/(x-a) = 0$. So

$$L = f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a};$$

the quantity $f'(a)$ is called the *derivative* of f at a. If f is differentiable at a, the line $y = f(a) + f'(a)(x-a)$ is the *tangent* of f at the point $(a, f(a))$. Alternative notation for the derivative is

$$\frac{d}{dx}f(x), \quad \frac{df(x)}{dx}, \quad \frac{df}{dx}, \quad \frac{dy}{dx}.$$

Rules for derivatives follow from theorems for limits:

$$\begin{aligned} (f+g)' &= f' + g', \\ (cf)' &= cf', \quad c \text{ constant} \\ (fg)' &= f'g + fg' \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \quad \text{if } g \neq 0 \end{aligned}$$

The chain rule. If the composition $f \circ g$ is defined, we have

$$(f \circ g)' = (f' \circ g) \cdot g' \quad \text{or} \quad \frac{d(f\circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

If f is differentiable at a, it is continuous at a, but not vice versa.

$$\lim_{x\to a}(f(x) - f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0.$$

To see that the continuity need not imply differentiability consider $f(x) = |x|$ which is continuous but not differentiable at $a = 0$ (a cusp).

In addition to a derivative at a point we define the *derivative from the right* and the *derivative from the left*:

$$f'_+(c) = \lim_{x\to c+}\frac{f(x)-f(c)}{x-c}, \quad f'_-(c) = \lim_{x\to c-}\frac{f(x)-f(c)}{x-c}.$$

The derivative $f'(c)$ exists if and only if both $f'_+(c)$ and $f'_-(c)$ exist and are equal.

Theorem 11 (Rolle's theorem)**.** *Let f be continuous on $[a,b]$ and differentiable on (a,b), and let $f(a) = f(b)$. Then there exists c between a and b such that $f'(c) = 0$.*

Proof given in the Lectures.

The proof of the following theorem appeared in the Gazette of the Australian Mathematical Society in 2007, given by a first year student Peng Zhang and your lecturer.

Theorem 12 (Cauchy's mean value theorem)**.** *Let functions $f, g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and let $g'(x) \neq 0$ on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Define $G(x) = f(x) - Ag(x)$ with the constant A chosen so that $G(a) = G(b)$. First we show that $g(a) \neq g(b)$. If we had $g(a) = g(b)$, hen by Rolle's theorem there would exist $d \in (a,b)$ such that $g'(d) = 0$; but his contradicts our assumption about the derivative of g. Hence $g(a) \neq g(b)$. Solving $G(a) = G(b)$ for A gives $A = (f(b) - f(a))/(g(b) - g(a))$. Applying Rolle's theorem yet again, this time to G, we conclude that there exists $c \in (a,b)$ such that $G'(c) = 0$, that is,

$$G'(c) = f'(c) - Ag'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(c) = 0;$$

the result then follows.

As a special case we obtain the so called (Lagrange's) mean value theorem.

Theorem 13 (Mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(a).$$

It follows from Cauchy's mean value theorem when we set $g(x) = x$.

Theorem 14 (l'Hôpital's rule 0/0)**.** *Suppose that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{7}$$

provided the limit on the right hand side exists.

Proof. We redefine the values of f and g at a by setting $f(a) = 0 = g(0)$; this does not have any effect on the hypotheses or the conclusion of the theorem as the limit at a does not depend on the functional value at a. Then f, g become continuous at a.

Let us first consider the limit from the right. If the limit on the right hand side in (7) exists as $x \to a+$, the derivatives $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$ for all x in some interval $(a, a+\delta)$. For any $x \in (a, a+\delta)$ the functions f and g are continuous in $[a,x]$ and differentiable in (a,x). By Theorem 12 there exists $c(x) \in (a,x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c(x))}{g'(c(x))}.$$

Since $a < c(x) < x$, $c(x) \to a$ as $x \to a+$, and (7) is true for $x \to a+$. The case $x \to a-$ follows by symmetry. Combining the two cases we get the result for $x \to a$.

Theorem 15 (l'Hôpital's rule ∞/∞)**.** *Let* $\lim_{x\to a}|f(x)| = \infty = \lim_{x\to a}|g(x)|$. *Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{8}$$

provided the limit on the right hand side exists.

The proof of this result is technically more complicated, and we will not give it here. The limits involving $x \to a$ in Theorems 2 and 3 can be replaced by limits involving $x \to a+$, $x \to a-$, $x \to \infty$ and $x \to -\infty$; no proofs are given for these modifications.

We give several examples of the rules. You may have to continue using the rule several times until you obtain an expression whose limit you can calculate. Remember that at each step you must check that you have either type $0/0$ or ∞/∞; otherwise you get a nonsensical result. In some examples you need to use the interchange of the limit and function, such as $\lim_{x\to a} F(\varphi(x)) = F\left(\lim_{x\to a}\varphi(x)\right)$; this is justified only if F is continuous. It is interesting to observe how indeterminates of type 0^0, $\infty \cdot 0$, ∞^0 or $\infty - \infty$ can be transformed to types $0/0$ or ∞/∞.

Example 9. $\displaystyle\lim_{x\to 0}\frac{\sin x - x}{x^3}$ (type $0/0$)

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}.$$

Example 10. $\displaystyle\lim_{x\to 0+} x^x$ (type 0^0)

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+}\exp(x\log x) = \exp\left(\lim_{x\to 0+} x\log x\right) = \exp\left(\lim_{x\to 0+}\frac{\log x}{1/x}\right)$$

$$= \exp\left(\lim_{x\to 0+}\frac{1/x}{-1/x^2}\right) = \exp\left(\lim_{x\to 0+} -x\right) = \exp(0) = 1.$$

Example 11. $\displaystyle\lim_{x\to\infty} xe^{-x}$ (type $\infty \cdot 0$)

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = \lim_{x\to\infty} e^{-x} = 0.$$

Example 12. $\displaystyle\lim_{x\to\frac{\pi}{2}-}(\tan x)^{\sin 2x}$ (type ∞^0)

$$\lim_{x\to\frac{\pi}{2}-}(\tan x)^{\sin 2x} = \lim_{x\to\frac{\pi}{2}-}\exp(\sin 2x\log\tan x) = \exp\left(\lim_{x\to\frac{\pi}{2}-}\sin 2x\log\tan x\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{2}-}\frac{\log\tan x}{\mathrm{cosec} 2x}\right) = \exp\left(\lim_{x\to\frac{\pi}{2}-}\frac{(1/\tan x)\sec^2 x}{-2\mathrm{cosec}^2 2x\cos 2x}\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{2}-} -\frac{\sin^2 2x}{2\tan x\cos^2 x\cos 2x}\right) = \exp\left(\lim_{x\to\frac{\pi}{2}-}\frac{\sin^2 2x}{\sin 2x\cos 2x}\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{2}-}\tan 2x\right) = \exp(0) = 1.$$

Example 13. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ (type $\infty - \infty$)

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \lim_{x\to 0}\frac{x^{-2}\sin^2 x - 1}{\sin^2 x} = \lim_{x\to 0}\frac{2\sin x\cos x\cdot x^{-2} - 2\sin^2 x\cdot x^{-3}}{2\sin x\cos x}$$

$$= \lim_{x\to 0}\frac{1}{\cos x}\lim_{x\to 0}\frac{x\cos x - \sin x}{x^3} = \lim_{x\to 0}\frac{-x\sin x}{3x^2} = -\frac{1}{3}\lim_{x\to 0}\frac{\sin x}{x}$$

$$= -\frac{1}{3}$$

Differentiation of the inverse function. If $f\colon I \to J$, where I and J are open intervals, and f is a differentiable bijection with $f'(x_0) \neq 0$, then the inverse function $g\colon J \to I$ is also differentiable and

$$g'(y_0) = \frac{1}{f'(g(y_0))} \quad \text{for } y_0 = f(x_0) \in J.$$

This is proved as follows: We have

$$y = f(x),\ x \in I \quad \Leftrightarrow \quad x = g(y),\ y \in J.$$

If $y_0 \in J$, then

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_o}{f(x) - f(x_0)} = \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}}.$$

We note that $x \to x_0 \Leftrightarrow y \to y_0$. Taking the limit as $y \to y_0$ gives the required formula. The condition $f'(x_0) \neq 0$ cannot be omitted: Consider the differentiable bijection $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$, $x \in \mathbb{R}$ at the point $x_0 = 0$.

Picard's iterative method. This is a method for the solution of the equation $x = f(x)$, where $f\colon [a, b] \to [a, b]$. A self-map f of $[a, b]$ is a *contraction* if there exists a constant α, $0 < \alpha < 1$, such that $|f(v) - f(u)| \leq \alpha|u - v|$ for all $u, v \in [a, b]$; α is called a *coefficient of contraction* for f on $[a, b]$. If the function f is differentiable on $[a, b]$ and

$$\alpha := \sup_{a \leq x \leq b} |f'(x)| < 1,$$

an application of the MVT shows that f is a contraction on $[a, b]$.

Theorem 16. *Let $f\colon [a, b] \to \mathbb{R}$ be a self-map of $[a, b]$ and a contraction. Then the equation $x = f(x)$ has a unique solution $x^* = f(x^*)$ obtained as the limit of the Picard iteration sequence $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$ for an arbitrary initial approximation $x_0 \in [a, b]$.*

This follows from Problem 36.

Newton's iterative method. We wish to solve the equation $F(x) = 0$, where F is a real valued differentiable function. Starting with an estimate x_0 for the solution such that $F'(x_0) \neq 0$, we attempt to improve it by iterations. If x_n is found and $F'(x_n) \neq 0$, we form a linear approximation to $F(x)$,

$$L(x) = F(x_n) + F'(x_n)(x - x_n),$$

and find the solution x_{n+1} to $L(x) = 0$:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{9}$$

If such a sequence converges, the limit is a solution to $F(x) = 0$ (check). The terms x_n of the sequence (9) are called the *Newton iterations*.

Theorem 17. *Let $F\colon [a,b] \to \mathbb{R}$ be differentiable in $[a,b]$ with $F'(x) \neq 0$ for all x, and let*

$$x - \frac{F(x)}{F'(x)} \in [a,b] \text{ for all } x \in [a,b].$$

If the derivative F' is continuous on $[a,b]$ and differentiable on (a,b), and if

$$\alpha := \sup_{a \le x \le b} \left| \frac{F(x)F''(x)}{[F'(x)]^2} \right| < 1,$$

then the Newton iterations converge to the unique solution $x^* \in [a,b]$ of $F(x) = 0$.

Proof. Set $f(x) := x - F(x)/F'(x)$ on $[a,b]$. We prove that $f(x)$ is a self-map of $[a,b]$ and α as defined above is a contractive constant for f on $[a,b]$, and apply Problem 36. The fixed point of $f(x)$ is a solution for $F(x) = 0$.

Riemann integral. Let $I = [a,b]$ be a closed bounded interval in $\mathbb{R}$. A *partition* $\mathcal{P}$ of I is given as $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The *norm* of this partition is the number $\|\mathcal{P}\| = \max_{1 \le k \le n} |x_k - x_{k-1}|$. A *tagged partition* of $[a,b]$ is a partition $\overline{\mathcal{P}}$ such that for each k a point $t_k \in [x_{k-1}, x_k]$ is chosen. If $\overline{\mathcal{P}}$ is a tagged partition of $[a,b]$, we define the *Riemann sum* of $f\colon [a,b] \to \mathbb{R}$ corresponding to $\overline{\mathcal{P}}$ as the number

$$S(f; \overline{\mathcal{P}}) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}).$$

A function $f\colon [a,b] \to \mathbb{R}$ is *Riemann integrable* (or *R-integrable* for short) if there exists a number $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any tagged partition $\overline{\mathcal{P}}$ satisfying $\|\overline{\mathcal{P}}\| < \delta$ we have

$$|S(f; \overline{\mathcal{P}}) - L| < \varepsilon.$$

It can be proved that if f is Riemann integrable, then the number L in the definition is unique; L is called the *Riemann integral* of f over $[a,b]$. The definition does not tell us how to calculate the Riemann integral if it exists. For this we need to explore further properties of the integral. The usual notation for the Riemann integral is

$$\int_a^b f(x)\,dx \quad \text{or simply} \quad \int_a^b f.$$

We also make the following convention: If $b < a$, we define

$$\int_a^b f := -\int_b^a f, \qquad \int_a^a f := 0.$$

Some properties of the Riemann integral. If f, g are Riemann integrable, then:

(i) $\displaystyle\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ (*linear*).

(ii) If $f \le g$ on $[a,b]$, then $\displaystyle\int_a^b f \le \int_a^b g$ (*monotone*).

(iii) Every Riemann integrable function is bounded. (A proof by contradiction.)

(iv) Let $f\colon [a,b] \to \mathbb{R}$, and let $a < c < b$. If f is R-integrable on $[a,b]$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

If f is R-integrable on $[a,c]$ and on $[c,b]$, then it is R-integrable on $[a,b]$ and the addition formula holds.

Example 14. A function with no Riemann integral. Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational}, \\ 1, & x \text{ rational}. \end{cases}$$

Let $\delta > 0$. Then we can choose a tagged partition $\overline{\mathcal{P}}$ and a tagged partition $\overline{\mathcal{Q}}$ such that $S(f,\overline{\mathcal{P}}) = 1$ and $S(f,\overline{\mathcal{Q}}) = 0$, while both partition have norm less than δ.

The following criterion of R-integrability is very useful in practice as we do not have to know the value of the integral beforehand, and as it deals with partitions which have *the same division points.*

Theorem 18 (Cauchy criterion)**.** *A bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any two tagged partitions with the same division points $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$ of norm less than δ satisfy*

$$|S(f;\overline{\mathcal{P}}) - S(f;\overline{\mathcal{Q}})| < \varepsilon.$$

Theorem 19. *Every function f continouous on a closed bounded interval $[a,b]$ is Riemann integrable.*

Proof is based on the Cauchy criterion and on the fact that a function continuous on a closed bounded interval is uniformly continuous on this interval.

Theorem 20. *Every function f monotonic on a closed bounded interval $[a,b]$ is Riemann integrable.*

Proof is again based on the Cauchy criterion.

Let I be a real interval. A function $F\colon I \to \mathbb{R}$ is a *primitive* to a function $f\colon I \to \mathbb{R}$ on I if $F'(x) = f(x)$ for all $x \in I$. A function $G\colon [a,b] \to \mathbb{R}$ is a *generalized primitive* to f on I if it is continuous on I and if there is a finite set $E \subset I$ such that $G'(x) = f(x)$ for all $\in I \setminus E$. The continuity of F is crucial:

Theorem 21. *Let F, G be two generalized primitives to a function $f\colon [a,b] \to \mathbb{R}$. Then there is a constant C such that $F(x) - G(x) = C$ for all $x \in [a,b]$.*

Proof. Combining the sets of which the derivatives of F and G do not exist, we may assume that there is a partition $a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$ such that $H := F - G$ is differentiable on each interval (s_{k-1}, s_k) with $H'(x) = 0$. By the MVT, $H(x) = C_k$ for all $x \in (s_{k-1}, s_k)$, $k = 1, \ldots, n$, where C_k is a constant. By the continuity of H, $C_k = H(s_k-) = H(s_k+) = C_{k+1}$ for all $k = 1, \ldots, n-1$. So $C_1 = \cdots = C_n = C$.

Theorem 22 (Fundamental theorem of calculus I)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable, and let $F\colon [a,b] \to \mathbb{R}$ be a generalized primitive to f on $[a,b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof. We first prove the theorem in the case that F is a primitive to f, that is, $F'(x) = f(x)$ for all $x \in [a,b]$.

Let $\varepsilon > 0$ be given. Then there exists $\delta = \delta(\varepsilon) > 0$ such that for any tagged partition $\overline{\mathcal{P}}$ with $\|\overline{\mathcal{P}}\| < \delta$ we have

$$\left| S(f;\overline{\mathcal{P}}) - \int_a^b f \right| < \varepsilon.$$

Apply the MVT to F and the subintervals of $\overline{\mathcal{P}}$: There exist points $u_k \in [x_{k-1}, x_k]$ such that

$$F(x_k) - F(x_{k-1}) = F'(u_k)(x_k - x_{k-1}), \quad k = 1, \dots, n.$$

Hence

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1}) = \sum_{k=1}^{n} f(u_k)(x_k - x_{k-1}) = S(f;\overline{\mathcal{P}}(u_1, \dots, u_n)),$$

and

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the equality holds.

The case when F is a proper generalized primitive of f is left as an exercise.

Theorem 23 (Fundamental theorem of calculus II)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable, and let $F\colon [a,b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f$ for $a \le x \le b$. Then F is continuous on $[a,b]$. If f is continuous at the point $c \in [a,b]$, then $F'(c) = f(c)$.*

Proof. Continuity: Let $c \in [a,b]$. Then

$$F(x) - F(c) = \int_a^x f - \int_a^c f = \int_a^c f + \int_c^x f - \int_a^c f = \int_c^x f.$$

We know that f is bounded on $[a,b]$, say $|f(x)| \le M$ if $x \in [a,b]$. Thus

$$|F(x) - F(c)| = \left| \int_c^x f \right| \le \left| \int_c^x |f| \right| \le M|x - c|.$$

This proves that F is continuous at c. (The seemingly superflous absolute value sign is needed because we may have $x < c$. Note also that F is continuous at c whether or not f is continuous at c.)

Derivative: Supose that $c \in [a,b)$ and consider the right derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there is $\eta > 0$ such that

$$c \le x \le c + \eta \Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon.$$

For $h > 0$ sufficiently small, $F(c+h) - F(c) = \int_c^{c+h} f$, and

$$(f(c) - \varepsilon)h \le F(c+h) - F(c) = \int_c^{c+h} f \le (f(c) + \varepsilon)h.$$

This implies

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \le \varepsilon.$$

We have proved

$$\lim_{h \to 0+} \frac{F(c+h) - F(c)}{h} = f(c).$$

The assumptions of the following theorem are quite stringent to ensure that the Riemann integrals exist.

Theorem 24 (Substitution)**.** *Let $\varphi\colon [\alpha, \beta] \to \mathbb{R}$ have a continuous derivative on $[\alpha, \beta]$, and let $f\colon I \to \mathbb{R}$ be continuous on an interval I containing $\varphi([\alpha, \beta])$. Then*

$$\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx.$$

In some situations we may have to use the following version of the substitution theorem.

Theorem 25 (Substitution II)**.** *Suppose that in addition to the hypotheses of Theorem 24 we have $\varphi'(t) \neq 0$ for all $t \in [\alpha, \beta]$. Let ψ be the inverse function to φ. Then*

$$\int_\alpha^\beta f(\varphi(t))\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\psi'(x)\, dx.$$

A set $A \subset \mathbb{R}$ is called a *null set* if for each $\varepsilon > 0$ there exists a sequence (a_n, b_n) of open intervals such that $A \subset \bigcup_{n=1}^\infty (a_n, b_n)$ and $\sum_{n=1}^\infty (b_n - a_n) < \varepsilon$. (Anticipating a later topic, we define $\sum_{n=1}^\infty c_n := \lim_{n \to \infty}(c_1 + c_2 + \cdots + c_n)$.) Any finite or countably infinite set is null.

Theorem 26 (Lebesgue's criterion)**.** *A function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and continuous on $[a, b] \setminus A$, where A is a null set.*

Proof of this theorem is fairly technical, and will not be given. However, it is an important result to be remembered.

Example 15 (Thomae's function)**.** Let $f\colon [0, 1] \to \mathbb{R}$ be defined as follows. For each irrational number x in $[0, 1]$ we set $f(x) = 0$. Each rational number $r > 0$ in $[0, 1]$ can be written as $r = m/n$ with natural numbers m, n having no common divisors other than 1; we set $f(r) = f(m/n) = 1/n$. (Make a sketch.) The function f is continuous at each irrational point $x \in [0, 1]$ (Problem 53). Since the rational numbers in $[0, 1]$ form a null set, Thomae's function is Riemann integrable by Lebesgue's criterion with integral 0.

Theorem 27 (Composition theorem)**.** *If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, $\varphi\colon [c,d] \to \mathbb{R}$ continuous, and $f([a,b]) \subset [c,d]$, then the composition function $\varphi \circ f$ is Riemann integrable on $[a,b]$.*

Proof based on Lebesgue's criterion.

Theorem 28. *If f is Riemann integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Proof. Composition theorem with $\varphi(t) = |t|$.

Theorem 29 (Product theorem)**.** *If f, g are Riemann integrable on $[a,b]$, then so is the product fg.*

Proof. By the Composition theorem, f^2, g^2 and $(f+g)^2$ are all Riemann integrable. The result follows as we can write $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$.

Numerical integration.

Trapezoidal rule. This method is based on approximation of the continuous function $f\colon [a,b] \to \mathbb{R}$ by a *polygonal function* (a piecewise linear continuous function) using equidistant partitions. Let $n \in \mathbb{N}$ and let $h_n = (b-a)/n$. By $\mathcal{P}_n$ we denote the partition which divides $[a,b]$ into n equal subintervals each of length h_n. Define $x_k = a + kh_n$ and $y_k = f(x_k)$, $k = 0, 1, \dots, n$. We approximate f by a polygonal function with vertices at the points (x_k, y_k), $k = 0, 1, \dots, n$, and the integral of f on the subinterval $[x_k, x_{k+1}]$ by the area of a trapezoid. (The area of a trapezoid with the base h and the sides l_1, l_2 is given by $\frac{1}{2}h(l_1 + l_2)$.) Hence

$$T_n(f) = h_n \left[\tfrac{1}{2}y_0 + \sum_{k=1}^{n} y_k + \tfrac{1}{2}y_n\right]. \tag{10}$$

Theorem 30 (Error estimate for trapezoidal rule)**.** *Let f, f' and f'' be continuous on $[a,b]$ and let M be an uper bound for $|f''(x)|$ on $[a,b]$. Then*

$$\left|T_n(f) - \int_a^b f\right| \le \frac{(b-a)h_n^2}{12} \cdot M = \frac{(b-a)^3}{12n^2} \cdot M. \tag{11}$$

Proof. Let $y = Ax + B$ be the straight line approximating $y = f(x)$ in the first interval $[x_0, x_1]$; then $A = y_0$ and $B = (y_1 - y_0)/h_n$. Set $g(x) := f(x) - (A + Bx)$. Then g is twice differentiable with $g''(x) = f''(x)$, and $g(x_0) = g(x_1) = 0$. Two integrations by parts give

$$\begin{aligned}\int_{x_0}^{x_1} (x-x_0)(x-x_1)f''(x)\,dx &= \int_{x_0}^{x_1} (x-x_0)(x-x_1)g''(x)\,dx \\ &= -2\int_{x_0}^{x_1} g(x)\,dx = -2\left(\int_{x_0}^{x_1} f(x)\,dx - h_n\frac{y_0+y_1}{2}\right).\end{aligned}$$

By the absolute value estimate for the integral,

$$\begin{aligned}\left|\int_{x_0}^{x_1} f(x)\,dx - h_n\frac{y_0+y_1}{2}\right| &\le \frac{1}{2}\int_{x_0}^{x_1}(x-x_0)(x_1-x)|f''(x)|\,dx\\ &\le \frac{M}{2}\int_{x_0}^{x_1}(-x^2+(x_0+x_1)x-x_0x_1)\,dx\\ &= \frac{M}{12}(x_1-x_0)^3 = \frac{M}{12}h_n^3.\end{aligned}$$

A similar estimate holds on each subinterval $[x_{k-1}, x_k]$ for $k = 1, \dots, n$. Therefore

$$\begin{aligned}\left|\int_a^b f(x)\,dx - T_n(f)\right| &= \left|\sum_{k=1}^{n}\left(\int_{x_{k-1}}^{x_k} f(x)\,dx - h_n\frac{y_{k-1}+y_k}{2}\right)\right|\\ &\le \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k} f(x)\,dx - h_n\frac{y_{k-1}+y_k}{2}\right|\\ &\le \sum_{k=1}^{n}\frac{M}{12}h_n^2 = \frac{M}{12}nh_n^3 = \frac{M(b-a)}{12}h_n^2.\end{aligned}$$

Simpson's rule. This method gives a considerably better approximation than the trapezoidal rule. Instead of approximating the function by polygonal functions, it is approximated by parabolas. The interval $[a, b]$ must be divided into an *even* number n of subintervals. Set $h_n = (b-a)/n$, $x_k = a + kh_n$ and $y_k = f(x_k)$, $k = 0, 1, \dots, n$; then $x_0 = a$ and $x_n = b$. On each of the 'double' adjacent intervals $[x_0, x_2]$, $[x_2, x_4]$, …, $[x_{n-2}, x_n]$ the continuous function f is approximated by a parabola; for the subinterval $[x_{2m}, x_{2m+2}]$ the parabola passes passes through the points (x_{2m}, y_{2m}), (x_{2m+1}, y_{2m+1}), (x_{2m+2}, y_{2m+2}), $m = 0, 1, \dots, \frac{1}{2}(n-2)$. Approximating the integral $\int_{x_{2m}}^{x_{2m+2}} f$ by the integral of the corresponding parabola and adding them all up, we obtain

$$\begin{aligned}S_n(f) &= \tfrac{1}{3}h_n[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]\\ &= \tfrac{1}{3}h_n(y_{\text{ends}} + 4y_{\text{odds}} + 2y_{\text{evens}}).\end{aligned} \tag{12}$$

Obtaining an error estimate for Simpson's rule is more difficult than it was for the Trapezoidal rule. Proofs can be found in textbooks on numerical analysis.

Theorem 31 (Error estimate for Simpson's rule)**.** *Let $f^{(j)}$, $j = 0, 1, 2, 3, 4$, be continuous on $[a, b]$ and let M be an uper bound for $|f^{(4)}(x)|$ on $[a, b]$. Then*

$$\left|S_n(f) - \int_a^b f\right| \le \frac{M(b-a)h_n^4}{180} = \frac{M(b-a)^5}{180n^4}. \tag{13}$$

Integrals dependent on a parameter. Let $f(x, t)$ be a function of two variables, $f\colon [a, b] \times [u, v] \to \mathbb{R}$. We say that f is *continuous at the point* $(x_0, t_0) \in \mathbb{R}^2$ if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|x - x_0| < \delta(\varepsilon) \text{ and } |t - t_0| < \delta(\varepsilon) \Rightarrow |f(x, t) - f(x_0, t_0)| < \varepsilon. \tag{14}$$

Since the rectangle $Q := [a,b] \times [u,v]$ is closed and bounded in $\mathbb{R}^2$, any function f continuous on Q is *uniformly continuous* on Q. This means that for each $\varepsilon > 0$ there exists a uniform $\delta(\varepsilon) > 0$, dependent on ε but not on the point $(x_0, t_0) \in Q$. A proof can be constructed along the same lines as the proof of the uniform continuity on a closed bounded subset of $\mathbb{R}$.

We consider integrals of the form

$$F(t) := \int_a^b f(x,t)\,dx, \tag{15}$$

where $t \in [u,v]$ is a parameter; the integral defines a function $F\colon [u,v] \to \mathbb{R}$. We are concerned with the continuity and differentiability of F.

COMMENT on continuity in two variables. The *joint* continuity in (x,y) is different from the *separate* continuity in x and in y. This is probably best demonstrated on an example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ \\ 0, & (x,y) = (0,0). \end{cases}$$

We show that f is separately continuous at $(0,0)$, but not jointly continuous there. Separate continuity at $(0,0)$: We approach the point first along the x-axis, and then along the y-axis:

$$\lim_{x\to 0} f(x,0) = 0 = f(0,0), \qquad \lim_{y\to 0} f(0,y) = 0 = f(0,0).$$

In fact, we can approach $(0,0)$ along any line $y = kx$, $k \neq 0$, and get directional continuity:

$$\lim_{x\to 0} f(x,kx) = \lim_{x\to 0} \frac{kx^3}{x^4 + k^2x^2} = \lim_{x\to 0} \frac{kx}{x^2 + k^2} = 0 = f(0,0).$$

If f were jointly continuous at $(0,0)$, we would be able to approach $(0,0)$ along any curve, and get the limit 0. However, this is not the case. If we approach along the parabola $y = x^2$, we no longer get 0:

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0.$$

This shows that f is not (jointly) continuous at $(0,0)$. (The word 'joint' is usually omitted in the case of functions of two or more variables.)

Here is an example how to justify the continuity of a function of two variables: Let $f(x,y) = \exp(2xy^3 + \log(1+x^2))$; this function is defined for all $(x,y) \in \mathbb{R}^2$. We use theorems on continuity. Define

$$f_1(x,y) = x, \quad f_2(x,y) = y, \quad f_3(u) = \log u, \quad f_4(v) = \exp v.$$

We have this useful principle $P(1 \to 2)$: If $\varphi(x)$ is continuous as a function of one variable, then $\Phi(x,y) = \varphi(x)$ is (jointly) continuous as a function of two variables. So: f_1 and f_2 are continuous by $P(1 \to 2)$, f_3 and f_4 are continuous as functions of one variable, and

$$f = f_4 \circ [2f_1 f_2^3 + f_3 \circ (1 + f_1^2)]$$

is continuous by the composition rule, the product rule and the sum rule.

Theorem 32 (Continuity and differentiation under the integral sign)**.** (a) *Let f be continuous in (x,t) on $Q := [a,b] \times [u,v]$. Then the function F defined in* (15) *is continuous on $[u,v]$, and*

$$\lim_{t \to t_0} \int_a^b f(x,t)\,dt = \int_a^b \lim_{t \to t_0} f(x,t)\,dx. \tag{16}$$

(b) *Let, in addition, $\partial f/\partial t$ be continuous on Q. Then F is differentiable on (u,v), and*

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx. \tag{17}$$

Proof. (a) Since f is uniformly continuous on Q, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(x,t) - f(x,t_0| < \varepsilon$ whenever $|t - t_0| < \delta(\varepsilon)$ (by (14) with $x = x_0$.) If $|t - t_0| < \delta(\varepsilon)$,

$$|F(t) - F(t_0)| \le \int_a^b |f(x,t) - f(x,t_0)|\,dx \le \int_a^b \varepsilon\,dx = \varepsilon(b-a).$$

This proves (16).

(b) By the mean value theorem, for any two points (x,t) and (x,t_0) there is a point $\xi(x,t)$ between t and t_0 such that

$$\frac{f(x,t) - f(x,t_0)}{t - t_0} = \frac{\partial f}{\partial t}(x, \xi(x,t)).$$

Hence

$$\frac{F(t) - F(t_0)}{t - t_0} = \int_a^b \frac{f(x,t) - f(x,t_0)}{t - t_0}\,dx = \int_a^b \frac{\partial f}{\partial t}(x,\xi(x,t))\,dx.$$

Since $\partial f/\partial t$ is continuous on Q, we can apply part (a) of this proof to obtain

$$\left.\frac{d}{dt}\right|_{t_0} \int_a^b f(x,t)\,dx = \lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0} = \lim_{t \to t_0} \int_a^b \frac{f(x,t) - f(x,t_0)}{t - t_0}\,dx$$

$$= \lim_{t \to t_0} \int_a^b \frac{\partial f}{\partial t}(x,\xi(x,t))\,dx \underset{\text{(a)}}{=} \int_a^b \lim_{t \to t_0} \frac{\partial f}{\partial t}(x,\xi(x,t))\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t_0)\,dx.$$

observing that $\lim_{t \to t_0} \xi(x,t) = t_0$ for any $x \in [a,b]$.

Example 16. Evaluate the integral $\displaystyle\int_0^1 \frac{x-1}{\log x}\,dx$. The integral has the appearance of an improper integral, but the integrand $g(x) = (x-1)/\log x$ can be made continuous on $[0,1]$ if we set

$$g(0) := \lim_{x \to 0+} g(x) = 0, \qquad g(1) := \lim_{x \to 1-} g(x) = 1.$$

(The second limit by l'Hôpital's rule.) We cannot evaluate the given integral directly, so we introduce a parameter t into the integrand to make it tractable:

$$F(t) := \int_0^1 \frac{x^t - 1}{\log x}\,dx.$$

We can check that the function $f(x,t) = (x^t - 1)/\log x$ is continuous on any rectangle $[0,1] \times [t_1, t_2]$, where $0 < t_1 < 1 < t_2$ if we set $f(0,t) = 0$ and $f(1,t) = t$. In calculating the derivative note that $x^t = \exp(t \log x)$:

$$\frac{\partial g(x,t)}{\partial t} = \frac{\partial}{\partial t} \frac{x^t - 1}{\log x} = \frac{x^t \log x}{\log x} = x^t$$

is also continuous on the rectangle $[0,1] \times [t_1, t_2]$. Thus by Theorem 32(b),

$$F'(t) = \frac{d}{dt} \int_0^1 \frac{x^t - 1}{\log x}\, dx = \int_0^1 \frac{\partial}{\partial t} \frac{x^t - 1}{\log x}\, dx = \int_0^1 x^t\, dx = \left[\frac{x^{t+1}}{t+1} \right]_0^1 = \frac{1}{t+1},$$

$t_1 < t < t_2$. Integrating this equation with respect to t, we get

$$F(t) = \int \frac{dt}{t+1} = \log(t+1) + C$$

with some constant C. To find the value of C we apply Theorem 32(a):

$$C = \lim_{t \to 0} F(t) = \lim_{t \to 0} \int_0^1 \frac{x^t - 1}{\log x}\, dx = \int_0^1 \lim_{t \to 0} \frac{x^t - 1}{\log x}\, dx = 0.$$

Thus $F(t) = \log(t+1)$, and

$$\int_0^1 \frac{x - 1}{\log x}\, dx = F(1) = \log 2.$$

Improper integrals. Improper integrals come in two basic types: Improper integral of the first kind, where the domain of integration is an unbounded interval, and the improper integral of the second kind, where the domain of integration is a bounded interval, but the function is unbounded. Improper integrals can be a mixture of both types.

Improper integral of the first kind. Suppose that the function $f\colon [a, \infty) \to \mathbb{R}$ is Riemann integrable on each interval $[a, b]$, $a < b < \infty$. Then

$$\int_a^\infty f(x)\, dx := \lim_{b \to \infty} \int_a^b f(x)\, dx$$

provided the limit exists and is finite. Similarly is defined the integral

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx.$$

Example 17.

$$\int_0^\infty \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_0^b \frac{dx}{1 + x^2} = \lim_{b \to \infty} \arctan b = \tfrac{1}{2}\pi.$$

Example 18. Let $p > 1$. Then

$$\int_1^\infty \frac{dx}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = \frac{1}{p-1}, \quad p > 1.$$

Check that this integral diverges if $p \le 1$.

Example 19. For $a > 0$, $a^x = \exp(x\log a)$. If $0 < a < 1$, then $\log a < 0$, and $a^x \to 0$ as $x \to \infty$. Let $0 < a < 1$. Then

$$\int_1^\infty a^x\,dx = \lim_{b\to\infty}\int_1^b a^x\,dx = \lim_{b\to\infty}\frac{a^b - a}{\log a} = -\frac{a}{\log a};$$

this integral diverges if $a \ge 1$.

If a function $F\colon [a,\infty) \to \mathbb{R}$ is monotonic and bounded, then the limit $\lim_{t\to\infty} F(t)$ exists and is finite. This follows from the monotonic sequence theorem when we use Heine's sequential characterization of the limit of $F(t)$. This can be applied to improper integrals:

Theorem 33. *Let $f\colon [a,\infty) \to \mathbb{R}$ be a nonnegative function such that the Riemann integral $F(t) := \int_a^t f(x)\,dx$ exists for each $t > a$. If the function $F(t)$ is bounded on $[a,\infty)$, then the improper integral $\int_a^\infty f(x)\,dx$ exists.*

Proof. The function F is increasing: If $t_2 > t_1$, then

$$F(t_2) = \int_a^{t_1} f(x)\,dx + \int_{t_1}^{t_2} f(x)\,dx \ge \int_a^{t_1} f(x)\,dx = F(t_1).$$

It also follows that if F is unbounded, the improper integral diverges to ∞.

We say that an improper integral $\int_a^\infty f(x)\,dx$ is *absolutely convergent* if the integral $\int_a^\infty |f(x)|\,dx$ converges. We show that an absolutely convergent integral is convergent, that is, we show that

$$\lim_{b\to\infty}\int_a^b |f(x)|\,dx \text{ exists } \Rightarrow \lim_{b\to\infty}\int_a^b f(x)\,dx \text{ exists.}$$

For this we use Heine's sequential characterization of convergence. Write $F(t) := \int_a^t f(x)\,dx$ and $G(t) := \int_a^t |f(x)|\,dx$. By hypothesis, $\lim_{n\to\infty} G(t_n) = L$ exists (and is finite) for any sequence (t_n) with $t_n \nearrow \infty$. If $m > n$, then

$$\begin{aligned}|F(t_m) - F(t_n)| &= \left|\int_{t_n}^{t_m} f(x)\,dx\right| \le \int_{t_n}^{t_m} |f(x)|\,dx = G(t_m) - G(t_n)\\ &\le |G(t_m) - L| + |G(t_n) - L|;\end{aligned}$$

since $(G(t_n))$ is a convergent sequence with the limit L, this implies that $(F(t_n))$ is a Cauchy sequence, and therefore convergent. This proves that the limit $\lim_{t\to\infty} F(t)$ exists, and the improper integral $\int_a^\infty f(x)\,dx$ converges.

Example 20. We show that the integral

$$\int_1^\infty \frac{\sin x}{x^2}\,dx$$

converges. First we consider its absolute convergence:

$$\int_1^t \frac{|\sin x|}{x^2}\,dx \le \int_1^t \frac{dx}{x^2} = 1 - \frac{1}{t} \le 1 \text{ for all } t \ge 1.$$

By Theorem 33, the integral $\int_1^\infty (|\sin x|/x^2)\,dx$ converges. Hence also the integral $\int_1^\infty (\sin x/x^2)\,dx$ converges.

However, there are convergent improper integrals of the type $\int_a^\infty f(x)\,dx$ for which the improper integral of the absolute value of f does not converge. The following example exhibits such an integral.

Example 21. Consider the integral $\int_1^\infty (\cos x/x)\,dx$. Integrating by parts, we get

$$\int_1^t \frac{\cos x}{x}\,dx = \frac{\sin t}{t} - \sin 1 + \int_1^t \frac{\sin x}{x^2}\,dx;$$

this converges by Example 20 as $t \to \infty$. Hence

$$\int_1^\infty \frac{\cos x}{x}\,dx = -\sin 1 + \int_1^\infty \frac{\sin x}{x^2}\,dx.$$

Now we test the convergence of $\int_1^\infty (|\cos x|/x)\,dx$:

$$\int_\pi^{n\pi} \frac{|\cos x|}{x}\,dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\cos x|}{x}\,dx \ge \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\cos x|}{k\pi}\,dx = \sum_{k=2}^n \frac{2}{k\pi}.$$

In the Lab Class 1 you considered the sequence $A_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$;

$$\begin{aligned} A_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &\ge 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{n-1}}{2^n} = 1 + \frac{n}{2} \to \infty \text{ as } n \to \infty; \end{aligned}$$

this shows that (A_n) diverges to ∞; hence the improper integral $\int_1^\infty (|\cos x|/x)\,dx$ diverges. Alternatively, we show that $\sum_{k=2}^n (1/k) \ge \int_2^n (dx/x) = \log n - \log 2$.

If $\int_a^\infty f(x)\,dx$ converges and $\int_a^\infty |f(x)|\,dx$ diverges, we say that the integral is *conditionally convergent.*

The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined by

$$\int_{-\infty}^\infty f(x)\,dx := \lim_{t\to\infty} \int_0^t f(x)\,dx + \lim_{s\to-\infty} \int_s^0 f(x)\,dx$$

provided both limits exist. Instead of 0 we can use any real terminal a.

Note. Calculating the limit with symmetrical terminals

$$\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$$

can lead to incorrect results, such as the wrong conclusion that $\int_{-\infty}^{\infty} x\,dx = 0$.

Example 22. We have

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\,dx = \int_0^{\infty}\frac{dx}{1+x^2} + \int_{-\infty}^{0}\frac{dx}{1+x^2} = \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) = \pi.$$

Comparison test for improper integral of the first kind. *Let $f, g\colon [a,\infty)\to\mathbb{R}$ be integrable on each interval $[a,b]$, $a<b<\infty$, and let $|f(x)| = O(|g(x)|)$ as $x\to\infty$.*
(a) *If $\int_a^{\infty} g(x)\,dx$ is absolutely convergent, then so is $\int_a^{\infty} f(x)\,dx$.*
(b) *If $\int_a^{\infty} |f(x)|\,dx$ diverges, then so does $\int_a^{\infty} |g(x)|\,dx$.*

(a) By hypothesis, there is a constant $M>0$ such that $|f(x)|\le M|g(x)|$ for all x in some interval $[K,\infty)$, where $K>a$. The convergence of $\int_K^{\infty}|f(x)|\,dx$ then follows, and $\int_a^{\infty} = \int_a^K + \int_K^{\infty}$. (b) follows by contradiction.

A special case of this can be formulated as a limit comparison test.

Limit comparison test. *Let $f, g\colon [a,\infty)\to\mathbb{R}$ be integrable on each interval $[a,t]$, $t>a$, and let*

$$\lim_{t\to\infty}\frac{|f(x)|}{|g(x)|} = A, \quad A<\infty. \tag{18}$$

(a) *If $\int_a^{\infty} g(x)\,dx$ is absolutely convergent, then so is $\int_a^{\infty} f(x)\,dx$.*
(b) *If $\int_a^{\infty} |f(x)|\,dx$ diverges, then so does $\int_a^{\infty} |g(x)|\,dx$.*

Example 23. Let us check whether the integral

$$\int_3^{\infty}\frac{2^x+x}{3^x-5x}\,dx$$

converges.

The integrand $f(x)$ is continuous on $[3,\infty)$, and so it is Riemann integrable on each closed bounded interval $[3,b]$. We guess that $f(x)$ behaves like $g(x) = (2/3)^x$ as $x\to\infty$, and we know that the integral of $g(x)$ converges on $[3,\infty)$. We suspect that $f(x) = O(g(x))$ as $x\to\infty$, or maybe even $f(x)\asymp g(x)$. To test this we try the limit test:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{2^x+x}{3^x-5x}\Big(\frac{2}{3}\Big)^{-x} = \lim_{x\to\infty}\frac{1+x\,2^{-x}}{1-5x\,3^{-x}} = 1.$$

This confirms that $f(x)\asymp g(x)$ as $x\to\infty$, and the integral $\int_3^{\infty} f(x)\,dx$ converges.

Improper integral of the second kind. In this type of integral the domain is bounded, but the function is unbounded. Suppose $f\colon (a,b] \to \mathbb{R}$ is such that the integral $\int_s^b f(x)\,dx$ exists for each s satisfying $a < s < b$, but f is unbounded on $(a,b]$. The improper integral $\int_a^b f(x)\,dx$ is defined by

$$\int_{a+}^{b} f(x)\,dx := \lim_{s\to a+} \int_s^b f(x)\,dx$$

provided the limit exists. The point $x = ba$ is a *singularity* for $f(x)$. We also define the improper integral

$$\int_a^{b-} f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx$$

when $f(x)$ has a singularity at $x = b$.

Example 24. We have

$$\int_{0+}^{1} \frac{dx}{\sqrt{x}} = \lim_{s\to 0+} \int_s^1 \frac{dx}{\sqrt{x}} = \lim_{s\to 0+} (2\sqrt{1} - 2\sqrt{s}) = 2.$$

Example 25. This time they are two problem points. (Sketch the function.)

$$\begin{aligned}\int_{-1+}^{1-} \frac{dx}{\sqrt{1-x^2}} &= \lim_{s\to -1+} \int_s^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{t\to 1-} \int_0^t \frac{dx}{\sqrt{1-x^2}}\\ &= \lim_{s\to -1+} (\arcsin 0 - \arcsin s) + \lim_{t\to 1-} (\arcsin t - \arcsin 0)\\ &= -\arcsin(-1) + \arcsin 1 = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.\end{aligned}$$

Convention. From now on we will stop using the $a+$ and $b-$ notation in the terminals of improper integrals, writing the terminals simply as a and b. The student should inspect the integrand for singularities and determine whether the integral is improper.

Comparison test for improper integral of the second kind. *Let $f, g\colon [a,b) \to \mathbb{R}$ be integrable on each interval $[a,t]$, $a < t < b$, and let*

$$|f(x)| = O(|g(x)|) \quad \text{as } x \to b. \tag{19}$$

(a) *If $\int_a^b g(x)\,dx$ is absolutely convergent, then so is $\int_a^b f(x)\,dx$.*
(b) *If $\int_a^b |f(x)|\,dx$ diverges, then so does $\int_a^b |g(x)|\,dx$.*

To prove (a): By (19) there is a constant $M > 0$ such that $|f(x)| \le M|g(x)|$ for all x in some interval $[c,b)$, $a < c < b$. We have

$$F(t) := \int_c^t |f(x)|\,dx \le M \int_c^t |g(x)|\,dx.$$

Since $F(t)$ is increasing, $\lim_{t\to b} F(t)$ exists as $F(t)$ is bounded by $M\int_c^b |g(x)|\,dx$. Then

$$\int_a^b |f(x)|\,dx = \int_a^c |f(x)|\,dx + \int_c^b |f(x)|\,dx$$

exists. The rest follows from the properties of absolute convergence of improper integrals. Part (b) is proved by contradiction.

A special case of this is the following:

Limit comparison test. *Let $f, g\colon [a, b) \to \mathbb{R}$ be integrable on each interval $[a, t]$, $a < t < b$, and let*

$$\lim_{t\to b-} \frac{|f(x)|}{|g(x)|} = A, \quad A < \infty. \tag{20}$$

(a) *If $\int_a^b g(x)\,dx$ is absolutely convergent, then so is $\int_a^b f(x)\,dx$.*

(b) *If $\int_a^b |f(x)|\,dx$ diverges, then so does $\int_a^b |g(x)|\,dx$.*

Example 26. The integrals of the form

$$\int_0^1 \frac{dx}{x^p}$$

provide good gauge functions for the comparison test. If $p < 1$ (possibly $p \le 0$), then the integral converges:

$$\int_0^1 \frac{dx}{x^p} = \lim_{s\to 0+} \frac{dx}{x^p} = \lim_{s\to 0+} \left(\frac{1}{-p+1} - \frac{s^{-p+1}}{-p+1} \right) = \frac{1}{1-p}, \quad p < 1.$$

(Here $-p+1 > 0$, so that $\lim_{s\to 0+} s^{-p+1} = 0$.) If $p \ge 1$, the integral diverges (check).

Example 27. We show that the improper integral

$$\int_{0+}^1 \log x\,dx \tag{21}$$

converges by comparison of $f(x) = \log x$ with $g(x) = x^{-1/2}$, which is known to have a convergent integral on $(0, 1)$. Try the limit test using l'Hôpital's rule:

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\log x}{x^{-1/2}} = \lim_{x\to 0+} \frac{x^{-1}}{-(1/2)x^{-3/2}} = -2 \lim_{x\to 0+} x^{1/2} = 0;$$

the limit comparison test then applies. (Alternatively find a primitive for $\log x$.)

Example 28. Use the comparison test to show that

$$\int_0^1 \frac{\log x}{x}\,dx \tag{22}$$

diverges. We compare $f(x) = x^{-1}|\log x|$ with $g(x) = x^{-1}$ which is known to have a divergent integral on $(0, 1)$. The limit test is inconclusive. We note that $|\log x| \ge 1$ if $x \ge e^{-1}$. So

$$\frac{1}{x} \le \frac{|\log x|}{x} \quad \text{if } 0 < x \le e^{-1},$$

that is, $g(x) = O(f(x))$ as $x \to 0+$. If $\int_o^1 x^{-1}|\log x|\,dx$ converged, so would $\int_o^1 x^{-1}\,dx$, which is a contradiction.

Mixed improper integrals We may have integrals which are simultaneously improper integrals of the first and second kind, and may have more than one singularity.

Example 29. Consider

$$\int_0^\infty \frac{dx}{x^{3/4}|x-1|^{1/4}(1+x^2)^{1/8}}.$$

Split the integral as $\int_0^{1/2} f$, $\int_{1/2}^1 f$, $\int_1^2 f$ and $\int_2^\infty f$: the first three integrals are improper of the second kind with the singularities at $x=0$ and $x=1$, the fourth is improper of the first kind. On each of the open subintervals the integrand $f(x)$ is continuous. We use the comparison tests:

$$f(x) = \begin{cases} O\left(\frac{1}{x^{3/4}}\right), & x \to 0+, \\ O\left(\frac{1}{|x-1|^{1/4}}\right), & x \to 1, \text{ that is, } |x-1| \to 0, \\ O\left(\frac{1}{x^{5/4}}\right), & x \to \infty. \end{cases}$$

The first two relations ensure that the integral converges on $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $[1,2]$, the last guarantees the convergence on $[2, \infty)$. Overall, the integral converges on $[0, \infty)$.

The Beta and Gamma functions. For real numbers $p > 0$ and $q > 0$ we define the integral

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt.$$

If $p \geq 1$ and $q \geq 1$, the integral is an ordinary Riemann integral, otherwise it is a convergent improper integral. For this consider $\int_0^{1/2} Q(t)\,dt$ and $\int_{1/2}^1 Q(t)\,dt$ with the integrand $Q(t) := t^{p-1}(1-t)^{q-1}$. The function $B(p,q)$ is known as the Beta function.

For real numbers $s > 0$ the convergent improper integral

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\,dt$$

defines the Gamma function. We show that the Gamma function is a generalization of the factorial, namely

$$\Gamma(n) = (n-1)! \quad \text{for any } n \in \mathbb{N}.$$

For this we set

$$I_n := \int_0^\infty e^{-t} t^{n-1}\,dt, \quad n \in \mathbb{N},$$

and prove by induction that $I_n = (n-1)!$. First, $I_1 = 1 = 0!$ Next we show that if $I_n = (n-1)!$ then $I_{n+1} = n!$ Let $x > 0$. Integrating by parts, we get

$$\int_0^x e^{-t} t^n \, dt = -e^{-x} x^n + n \int_0^x e^{-t} t^{n-1} \, dt.$$

Using the fact that $e^{-x} x^n \to 0$ as $x \to \infty$ and the assumption that $\lim_{x\to\infty} \int_0^x e^{-t} t^{n-1} = (n-1)!$ we get

$$I_{n+1} = \lim_{x\to\infty} \int_0^x e^{-t} t^n \, dt = n \cdot (n-1)! = n!$$

Infinite series. An *infinite series*

$$\sum_{k=1}^{\infty} a_k \tag{23}$$

is a convenient way to write the sequence (s_n), where

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, \ldots \tag{24}$$

The real numbers a_k are the *terms of the series* (23), and the numbers s_n defined in (24) are the *partial sums* of the series (23).

Definition. The series (23) is *convergent* if the sequence (s_n) of its partial sums is convergent, and we define its *sum* by

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} s_n. \tag{25}$$

Thus the symbol $\sum_{k=1}^{\infty} a_k$ stands for the series with terms a_k—whether it converges or not—and in the case it does converge, it stands also for the sum $\lim_{n\to\infty} s_n$. A summation with a finite number of terms belongs to algebra, a 'summation' with an infinite number of terms belongs to analysis because it involves the limit process.

We often start a series from an index other than 1, for instance,

$$\sum_{k=2}^{\infty} \frac{1}{\log k};$$

had we started from $k = 1$, the first term would have been undefined. The index k in the series $\sum_{k=1}^{\infty} a_k$ can be replaced by other characters, such as $\sum_{n=1}^{\infty} a_n$ or $\sum_{j=1}^{\infty} a_j$.

Example 30. The *geometric series* is the series whose nth term is a^n for some real number a:

$$\sum_{n=0}^{\infty} a^n, \qquad a \in \mathbb{R}.$$

If $a \neq 1$, then the induction shows that for every $m \in \mathbb{N}$,

$$s_m := 1 + a + a^2 + \cdots + a^m = \frac{1 - a^{m+1}}{1 - a}.$$

If $|a| < 1$, then $a^{m+1} \to 0$ as $m \to \infty$, and the series converges:

$$\sum_{n=0}^{\infty} a^n = \lim_{m \to \infty} s_m = \lim_{m \to \infty} \frac{1 - a^{m+1}}{1 - a} = \frac{1}{1 - a}.$$

The series diverges if $|a| \geq 1$.

Example 31. We use improper integrals to show that the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges: We observe that on the interval $[k, k+1]$ we have $1/x \leq 1/k$, and so $\int_k^{k+1} (1/x)\, dx \leq \int_k^{k+1} (1/k)\, dx = 1/k$. Then

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{dx}{x} = \int_1^{n+1} \frac{dx}{x}.$$

Since the improper integral $\int_1^{\infty} (1/x)\, dx$ diverges to ∞, the sequence (s_n) is unbounded, and therefore divergent.

Note: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, then the sequence (s_n) of its partial sums is increasing. Hence (s_n) converges if and only if it is bounded. Therefore a positive term series $\sum_{n=1}^{\infty} a_n$ converges if and only if its partial sums have a finite upper bound.

Example 32. Let p be a real number, $p \geq 1$. The *p-harmonic series* is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

Since this is a positive term series, it converges if and only its partial sums are bounded. We have seen that for $p = 1$ the harmonic series diverges because it has unbounded partial sums. We show that for $p > 1$ the partial sums of the p-harmonic series are bounded by comparing it with a convergent improper integral: On the interval $[k-1, k]$, $k = 2, 3, \ldots$

$$\frac{1}{k^p} \leq \frac{1}{x^p} \implies \frac{1}{k^p} = \int_{k-1}^{k} \frac{1}{k^p}\, dx \leq \int_{k-1}^{k} \frac{dx}{x^p},$$

and

$$s_n = 1 + \sum_{k=2}^{n} \frac{1}{k^p} \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{dx}{x^p} = 1 + \int_1^n \frac{dx}{x^p}.$$

Since the improper integral converges when $p > 1$, the partial sums are bounded, and the series converges.

Example 33. *Exponential series* is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Since this is a positive term series, for its convergence we need to show that the partial sums are bounded:

$$\begin{aligned} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} &\leq 1 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n-1} \\ &= 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right) < 3, \qquad n = 1, 2, \dots \end{aligned}$$

Thus the series is convergent. Using the binomial theorem expansion and some algebra, we can show that

$$\left(1 + \frac{1}{n}\right)^n \leq \sum_{k=1}^{n} \frac{1}{k!} < \left(1 + \frac{1}{n}\right)^n + \frac{3}{2n}$$

which proves that the sum of the series is Euler's number $e = \lim_n (1 + n^{-1})^n$ (the sandwich rule).

Theorem 34 (Divergence test)**.** *If the series* $\sum_{n=1} a_n$ *converges, then* $a_n \to 0$. *Contrapositively: If* $a_n \not\to 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Let $\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} s_m = s$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

Observe that the condition $a_n \to 0$ does not guarantee that the series $\sum_m a_n$ converges: In the harmonic series we have $1/n \to 0$, but the series itself diverges.

Theorem 35 (Algebra of series)**.** *Suppose that* $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$. *Then*

(i) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$;

(ii) $\displaystyle\sum_{n=1}^{\infty} c a_n = cA$ *for any* $c \in \mathbb{R}$;

(iii) *If* $a_n \leq b_n$ *for all* n, *then* $A \leq B$.

Telescoping. Telescoping refers to the 'fold-up' procedure

$$\sum_{n=1}^{p} (a_n - a_{n+1}) = a_1 - a_{p+1}.$$

We use it to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

First note that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$ (partial fractions). Then

$$\sum_{n=1}^{p} \frac{1}{n(n+1)} = \sum_{n=1}^{p} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{p+1},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{p\to\infty} \sum_{n=1}^{p} \frac{1}{n(n+1)} = \lim_{p\to\infty} \left(1 - \frac{1}{p+1} \right) = 1.$$

There are many similarities between improper integrals of the first kind and series.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 36. *An absolutely convergent series is convergent.*

Proof. Write

$$s_n = \sum_{k=1}^{n} a_n, \qquad t_n = \sum_{k=1}^{n} |a_k|.$$

Since the series is absolutely convergent, the sequence (t_n) of partial sums converges, and therefore is Cauchy. We show that (s_n) is also Cauchy. This follows from the following estimate for $m > n$:

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| = t_m - t_n.$$

(Observe how much the proof resembles the argument for improper integrals.)

Tests for absolute convergence.

Comparison test. *Let* $|a_n| = O(|b_n|)$ *as* $n \to \infty$.
If $\sum_{n=1}^{\infty} b_n$ *is absolutely convergent, then so is* $\sum_{n=1}^{\infty} a_n$.
If $\sum_{n=1}^{\infty} |a_n|$ *diverges, then so does* $\sum_{n=1}^{\infty} |b_n|$.

The proof is similar to the one for improper integrals of the first kind.

Limit comparison test. *Let* $\lim_{n\to\infty} a_n/b_n = A$, *where* $A < \infty$.

(a) *If* $\sum_{n=1}^{\infty} b_n$ *is absolutely convergent, then so is* $\sum_{n=1}^{\infty} a_n$.
(b) *If* $A \neq 0$ *and* $\sum_{n=1}^{\infty} |a_n|$ *diverges, then so does* $\sum_{n=1}^{\infty} |b_n|$.

The existence of the finite nonzero limits guarantees that $|a_n| \asymp |b_n|$ as $n \to \infty$.

Example 34. Does the series $\sum_{k=3}^{\infty} \dfrac{1}{(3k^4 - 7k^2 + 5)^{1/3}}$ converge? Using the limit test or otherwise we show that

$$\frac{1}{(3k^4 - 7k^2 + 5)^{1/3}}, \asymp \frac{1}{k^{4/3}} \text{ as } k \to \infty,$$

and conclude that the series converges absolutely by comparison with a p-harmonic series for $p = 4/3$.

Ratio test. *If* $r := \limsup_n |a_{n+1}|/|a_n| < 1$*, the series* $\sum_{n=1}^\infty a_n$ *is absolutely convergent. If* $l := \liminf_n |a_{n+1}|/|a_n| > 1$*, then it is divergent. The test is inconclusive if* $l < 1 < r$*.*

Let $r < 1$; choose ρ such that $r < \rho < 1$. There exists $N \in \mathbb{N}$ such that $|a_{n+1}/a_n| \leq \rho$ for all $n \geq N$. Then $|a_{N+p}| = |a_{N+p}/a_{N+p-1}| \ldots |a_{N+1}/a_N| \, |a_N| \leq |a_N|\rho^p$. Writing $n = N + p$, we get $|a_n| \leq (|a_N|\rho^{-N})\rho^n$ for $n \geq N$, that is, $|a_n| = O(\rho^n)$ as $n \to \infty$. The divergence part is proved similarly. This test is often used when the limit of the ratio exists.

Root test. *Let* $r := \limsup_n |a_n|^{1/n}$*. If* $r < 1$*, the series* $\sum_{n=1}^\infty a_n$ *converges absolutely, if* $r > 1$*, the series diverges. The test is inconclusive if* $r = 1$*.*

Let $r < 1$. Again choose ρ such that $r < \rho < 1$. Then for some N, $|a_n|^{1/n} \leq \rho$ for all $n \geq N$, that is, $|a_n| = O(\rho^n)$ as $n \to \infty$. If $r > 1$, then $a_n \not\to 0$.

The following example shows that the root test is more effective than the ratio test.

Example 35. Consider the positive term series $\sum_{n=1}^\infty a_n$ where $a_n = (\frac{1}{2})^n$ if n is odd, and $a_n = (\frac{1}{3})^n$ if n is even. Then $a_{n+1}/a_n = \frac{1}{3}(\frac{2}{3})^n$ if n is odd, and $a_{n+1}/a_n = \frac{1}{2} \cdot (\frac{3}{2})^n$ if n is even. Hence $\liminf_n (a_{n+1}/a_N) < 1 < \limsup_n (a_{n+1}/a_n)$, and the test is inconclusive. On the other hand, $|a_n|^{1/n} = \frac{1}{2}$ is n is odd, and $|a_n|^{1/n} = \frac{1}{3}$ if n is even. Hence $\limsup_n |a_n|^{1/n} = \frac{1}{2} < 1$, and the root test applies to ensure that the series converges (absolutely).

We can state formally the test we have previously used to investigate the convergence of the generalized harmonic series. The proof follows the argument we used earlier.

Integral test. *Let* $f \colon [1, \infty) \to \mathbb{R}$ *be a nonnegative decreasing function. The series* $\sum_{n=1}^\infty f(n)$ *is convergent if and only if the improper integral* $\int_1^\infty f(x)\, dx$ *is convergent.*

We may recall that monotonic functions are Riemann integrable. We do not give a full proof of this test which relies on a similar procedure we used when dealing with Examples 31 and 32. This is a useful and often employed test. The integral test implies that the series

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n \log^p n}$$

converge if $p > 1$, and diverge if $0 < p \leq 1$.

Absolutely convergent series enjoy a number of properties which are not shared by series which do not converge absolutely. The following two theorems give an example.

Theorem 37 (Rearrangement of terms)**.** *Let* $\sum_{n=1}^\infty a_n$ *be an absolutely convergent* *series. If* $\pi\colon \mathbb{N}\to\mathbb{N}$ *is a bijection, then*

$$\sum_{n=1}^{\infty} a_{\pi(n)} = \sum_{n=1}^{\infty} a_n.$$

Proof. Write $s_n = a_1 + \cdots + a_n$, $\sigma_n = |a_1| + \cdots + |a_n|$, and $\tau_n = a_{\pi(1)} + \cdots + a_{\pi(n)}$. Let $s = \sum_{k=1}^{\infty} a_k$, that is, $s = \lim_{n\to\infty} s_n$. Let $\varepsilon > 0$ be given. There is an index N such that

$$|\sigma_{N+p} - \sigma_N| = |a_{N+1}| + \cdots + |a_{N+p}| < \varepsilon \tag{26}$$

for every integer $p \geq 1$. Choose m so that $1, 2, \ldots, N$ are among $\pi(1)$, $\pi(2)$, ..., $\pi(m)$. For any $n > m$, in the expression $\tau_n - s_n$ the terms $a_1, \ldots, a_N$ cancel as they appear in τ_n as well as in s_n. The difference $\tau_n - s_n$ is then the sum of finitely many of the terms $\pm a_{N+1}$, $\pm a_{N+2}$, ... According to (26), $|\tau_n - s_n| < \varepsilon$ whenever $n > m$. So $\tau_n - s_n \to 0$ and

$$\tau_n = s_n + (\tau_n - s_n) \to s + 0 = s.$$

Theorem 38 (Cauchy product)**.** *Let* $\sum_{n=0}^{\infty} a_n$*,* $\sum_{n=0}^{\infty} b_n$ *be convergent series, at least one of them absolutely convergent. If* $c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$*, then*

$$\sum_{n=0}^{\infty} c_k = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. This result is known as Merten's theorem. Assume that $\sum_n a_n$ is absolutely convergent, and $\sum_n b_n$ convergent (absolutely or non-absolutely). Set

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k.$$

Suppose $\sum_n a_n$ converges to A and $\sum_n b_n$ to B. Then

$$C_n = \sum_{i=0}^{n} c_i = \sum_{i=0}^{n}\sum_{k=0}^{i} a_k b_{i-k} = \sum_{i=0}^{n} B_i a_{n-i}$$

by rearrangement. So

$$C_n = \sum_{i=0}^{n} (B_i - B) a_{n-i} + B A_n.$$

We want to show that $C_n \to AB$. Let $\varepsilon > 0$ be given. Whenever $n > N$, we estimate

$$\begin{aligned} |C_n - AB| &= \left|\sum_{i=0}^{n} (B_i - B) a_{n-i} + B(A_n - A)\right| \\ &\leq \sum_{i=0}^{N-1} |B_i - B|\,|a_{n-i}| + \sum_{i=N}^{n} |B_i - B|\,|a_{n-i}| + |B|\,|A_n - A|. \end{aligned}$$

Choosing N appropriately, and then taking n sufficiently large, we can make each of the summands in the last expression less than $\frac{1}{3}\varepsilon$.

Note. It is crucial that one of the series is *absolutely* convergent. For a counterexample take

$$a_n = b_n = \frac{(-1)^n}{n+1},$$

where the series $\sum_{n=0}^{\infty}(-1)^n(1/(n+1))$ will be shown to be convergent. However, the Cauchy product is not convergent.

Conditionally convergent series. If the series $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not, we say that the series is *conditionally convergent.*

We start with the partial summation formula which is a basic tool for deriving tests for conditional convergence.

Theorem 39 (Partial summation)**.** *Let the series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be given, and let* $B_n = \sum_{k=1}^{n} b_k$*. Then*

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1}(a_k - a_{k+1})B_k + a_n B_n, \quad n \geq 2.$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= a_1 B_1 + a_2(B_2 - B_1) + \cdots + a_n(B_n - B_{n-1}) \\ &= (a_1 - a_2)B_1 + (a_2 - a_3)B_2 + \cdots + (a_{n-1} - a_n)B_{n-1} + a_n B_n.\end{aligned}$$

Theorem 40 (Dirichlet's test for conditional convergence)**.** *Let the series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be given with* (a_n) *monotonic and* $a_n \to 0$*, and let the partial sums* $B_n = \sum_{k=1}^{n} b_k$ *be bounded. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*

Proof. Let $M > 0$ be an upper bound for $|B_n|$, and assume $a_n \searrow 0$ for definiteness. Then

$$\sum_{k=1}^{n-1} |(a_k - a_{k+1})B_k| \leq M \sum_{k=1}^{n}(a_k - a_{k+1}) = M(a_1 - a_n) \leq M a_1.$$

Hence the series $\sum_{k=1}^{\infty}(a_k - a_{k+1})B_k$ converges absolutely with the sum, say, S. By the partial summation,

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1}(a_k - a_{k+1})B_k + a_n B_n \to S + 0 = S \text{ as } n \to \infty.$$

The following test can be proved in a similar way.

Theorem 41 (Abel's test for conditional convergence)**.** *Let the series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be given with* (a_n) *monotonic and bounded, and let* $\sum_{k=1}^{\infty} b_k$ *be convergent. Then the series* $\sum_{k=1}^{\infty} a_k b_k$ *is convergent.*

Proof. Since (a_k) is monotonic and bounded, it is convergent to a limit a. Assume that (a_k) is decreasing. Then $c_k := a_k - a$ is monotonically convergent to 0: $c_k \searrow 0$. Since the series $\sum_k b_k$ converges, its partial sums are bounded. We now apply Dirichlet's test to $\sum_k b_k$ and $\sum_k c_k$ to conclude that $\sum_k b_k c_k$ converges. Then

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} (a + c_k) b_k = \sum_{k=1}^{\infty} a b_k + \sum_{k=1}^{\infty} b_k c_k.$$

The Leibniz test. A special case of Theorem 40 with $b_n = (-1)^n$ is the *Leibniz test*: *If (a_n) is a monotonic sequence with $a_n \to 0$, then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.*

Example 36. We proved that the harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent to ∞. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges conditionally by the Leibniz test. We will show in Example 41 that the sum of this series is $\log 2$.

What happens to a conditionally convergent series under a rearrangement of terms? This is a rather curious result:

If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series and A any real number, then there exists a rearrangement of terms of the series so that the rearranged series converges to A.

Taylor's polynomials. If the function f has derivatives up to order n, we can approximate the function in a neighbourhood of some point a by a polynomial of degree n, known as the *nth Taylor polynomial* of f centred at a:

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned} \tag{27}$$

We observe that the derivatives of P_n at a up to order n agree with the derivatives of f at a:

$$P_n^{(k)}(a) = f^{(k)}(a) \quad \text{for } k = 0, 1, \ldots, n$$

If the function f has $n+1$ derivatives, we get the following approximation theorem, an extension of the Mean value theorem:

Theorem 42 (Taylor's theorem)**.** *Suppose f has derivatives up to order $n+1$ in some open interval I containing a. Then for each $x \in I$ there exists a point c between a and x such that*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(f; a, x) \tag{28}$$

where

$$R_n(f;a,x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad (29)$$

Proof. Keep x fixed and for any t between a and x define

$$F(t) = f(t) - P_n(t) - A(x)(t-a)^{n+1}, \text{ where } A(x) = (f(x) - P_n(x))/((x-a)^{n+1}),$$

and P_n is defined by (27). Then $F(x) = 0 = F(a)$. By Rolle's theorem there is c_1 between a and x such that $F'(c_1) = 0$. Since $f'(a) = P_n'(a)$, we have $F'(a) = 0$. By another application of Rolle's theorem we conclude that there is a point c_2 between a and c_1 with $F''(c_2) = 0$. Continuing this way we find that there is $c := c_{n+1}$ between a and c_n such that $F^{(n+1)}(c) = 0$. Since P_n is a polynomial of degree n, $P_n^{(n+1)}$ is identically zero, in particular $P_n^{(n+1)}(c) = 0$. Hence $f^{(n+1)}(c) = A(x)(n+1)!$ which gives the result.

The remainder $R_n(f;a,x)$ given in (29) is called the *Lagrange form of remainder*. A Taylor polynomial centred at 0 is sometimes called a *Maclaurin polynomial*.

Example 37. Write down the nth Taylor polynomial for $f(x) = e^x$ centred at 0. How many terms you need to be sure that the approximation to e is correct to three decimal places?

For every integer n we have $f^{(n)}(x) = e^x$, and $f^{(n)}(0) = 1$. Hence $P_n(x)$ given by (27) is equal to

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},$$

and

$$R_n(f;0,x) = \frac{e^c}{(n+1)!}x^{n+1}, \quad 0 < c < x.$$

To get an approximation to $e = f(1)$ correct to 3 decimal places we need to get the remainder less than 0.5×10^{-3} (not 10^{-3} in view of round-off error). We use the upper bound $e < 3$ and require

$$R_n < \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!} < 5 \times 10^{-4}.$$

We have $3/(7!) \approx 5.9 \times 10^{-4}$; then 7 terms will give us the desired approximation

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!} \approx 2.718$$

correct to three decimal places.

Example 38. Find the nth Taylor polynomial of the function f centred at $x = 0$:

$$\text{(i) } f(x) = \frac{1}{1-x}, \qquad \text{(i) } f(x) = \frac{1}{1+x}, \qquad \text{(iii) } f(x) = \frac{x}{1+x^2}.$$

We could calculate the derivatives of the given functions at 0, but we use a shortcut based on the use geometric series expansions assuming $|x| < 1$:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + \cdots + x^n + O(x^{n+1}) \quad \text{as } x \to 0.$$

Here the $O(x^{n+1})$ term is $x^{n+1}(1 + x + x^2 + \cdots) = x^{n+1}(1-x)^{-1}$.

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + O(x^{n+1}) \quad \text{as } x \to 0,$$

$$\frac{x}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k+1} = x - x^3 + x^5 - \cdots + (-1)^n x^{2n+1} + O(x^{2n+3}) \quad \text{as } x \to 0.$$

Power series. Let $x_0 \in \mathbb{R}$ be a given point. The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x \in \mathbb{R}, \tag{30}$$

is called a *power series with centre* x_0 *and coefficients* $a_n \in \mathbb{R}$.

Theorem 43 (Radius of convergence)**.** *For any power series* (30) *there are three possibilities:*

(i) *The series converges absolutely for all* $x \in \mathbb{R}$.

(ii) *The series converges only for* $x = x_0$.

(iii) *There exists a real number* $R > 0$ *such that the series converges absolutely whenever* $|x - x_0| < R$ *and diverges whenever* $|x - x_0| > R$.

Proof. In the proof we will assume for simplicity that $x_0 = 0$; the general case is obtained by the translation $x \mapsto x - x_0$. Define $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$, admiting $\alpha = \infty$ is the sequence is unbounded.

(i) Let $\alpha = 0$. Apply the root test: $\limsup_n |a_n x^n|^{1/n} = \limsup_n |a_n|^{1/n} |x| = 0$, that is, the series converges absolutely for all $x \in \mathbb{R}$.

(ii) Let $\alpha = \infty$, that is, let $|a_n|^{1/n}$ be unbounded. There is a strictly increasing sequence of integers k_n such that $|a_{k_n}|^{1/k_n} \to \infty$. Hence $|a_{k_n} x^{k_n}|^{1/k_n} = |a_{k_n}|^{1/k_n}| \, |x| \to \infty$, which means that $a_n x^n \not\to 0$, and the series diverges.

(iii) Let $0 < \alpha < \infty$. If x satisfies $0 < |x| < 1/\alpha$, then $\limsup_n |a_n x^n|^{1/n} = \alpha |x| < 1$, and $\sum_n |a_n x^n|$ converges. For z satisfying $|z| > 1/\alpha$ we get $\limsup_n |a_n x^n| > 1$, and the series diverges. Then $R := 1/\alpha$ has the required property.

The number R from the preceding theorem is called the *radius of convergence* of the series (30). For consistency we say that in the case (i) the radius of convergence is ∞, and in the case (ii) we say it is 0. In the case when $0 < R < \infty$, we have the so-called *Cauchy–Hadamard formula* obtained in the preceding proof:

$$R = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}. \tag{31}$$

In the last case the series may converge absolutely, conditionally or diverge for $x = x_0 + R$ or $x = x_0 - R$. The *interval of convergence* is the set of all x for which the power series converges.

Example 39. Consider the power series $\sum_{n=1}^{\infty} x^n/n$ with centre 0 and coefficients $a_n = 1/n$. Apply the ratio test for absolute convergence:

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{n+1}{n}|x| \to |x| \quad \text{as } n \to \infty.$$

Then the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. This shows that the radius of convergence of the series is $R = 1$. The end points: If $x = 1$, we get the harmonic series $\sum_{n=1}^{\infty}(1/n)$ which diverges, if $x = -1$, we get the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}(1/n)$ which converges conditionally. The interval of convergence is the semiclosed interval $[-1, 1)$.

Example 40. We give a list of a few simple power series with centre 0 along with their radius and interval of convergence.

Power series	Radius	Interval
$\sum_{n=1}^{\infty} n^n x^n$	0	$\{0\}$
$\sum_{n=1}^{\infty} x^n/n!$	∞	$(-\infty, \infty)$
$\sum_{n=1}^{\infty} x^n$	1	$(-1, 1)$
$\sum_{n=1}^{\infty} x^n/n^2$	1	$[-1, 1]$
$\sum_{n=1}^{\infty} x^n/n$	1	$[-1, 1)$
$\sum_{n=1}^{\infty} (-1)^n x^n/n$	1	$(-1, 1]$

Theorem 44. *If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is a power series with radius of convergence R and the sequence $(|a_n/a_{n+1}|)$ converges to a finite limit, then*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}. \tag{32}$$

Proof. By the ratio test for absolute convergence of series, $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges if

$$\lim_{n\to\infty} \frac{|a_{n+1}(x - x_0)^{n+1}|}{|a_n(x - x_0)^n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x - x_0| < 1,$$

that is, the series converges if $|x - x_0| < \lim_n |a_n/a_{n+1}|$; diverges if $|x - x_0| > \lim_n |a_n/a_{n+1}|$. Hence $R := \lim_n |a_n/a_{n+1}|$ is the radius of convergence for the series.

Algebraic operations with power series. It is convenient to formulate the theorems for power series with centre 0; the general case is obtained by a translation of the variable. Suppose $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are power series with radii of convergence R_1 and R_2, respectively. Then

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n, \quad R \geq \min(R_1, R_2),$$

$$c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c a_n x^n, \quad R' = R_1,$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad R'' \geq \min(R_1, R_2),$$

where R, R' and R'' refer to the radii of convergence of the series on the right. The last equation is known as the Cauchy product formula for power series.

Power series can be differentiated and integrated term by term without a change in the radius of convergence. This means that any function defined by a power series is differentiable and integrable within its radius of convergence R:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad |x| < R,$$

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad |x| < R.$$

Example 41. We use the geometric series expansion for $(1-x)^{-1}$ to obtain power series representations for $(1-x)^{-2}$, $(1-x)^{-3}$ and $\log(1+x)$.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1}, \quad |x| < 1,$$

$$\frac{1}{(1-x)^3} = \frac{d}{dx}\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}, \quad |x| < 1,$$

$$\log(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^n\right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad -1 < x \leq 1.$$

Taylor series. Let f be a function which has derivatives of all orders near a point a. For each $n \in \mathbb{N}$ we can define the nth Taylor polynomial $P_n(f; a)$, and we have

$$f(x) = P_n(f; a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(f; a, x);$$

the remainder R_n can be expressed by the Lagrange's formula

$$R_n(f;a,x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \quad c \text{ between } a \text{ and } x.$$

If the remainder converges to zero on some open interval I containing a, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad x \in I; \tag{33}$$

the series in this equation is the *Taylor series* for f in I. Such a situation arises, for instance, if there exists $\alpha > 0$ such that

$$|f^{(k)}(x)| \leq \alpha^k, \quad x \in I,\ k \in \mathbb{N}, \tag{34}$$

in view of the Lagrange's formula for the remainder. This is a very useful condition for the convergence of the Taylor series for f. A Taylor series centred at 0 is often called a *Maclaurin series.*

Example 42. We give examples of Taylor series of some elementary functions.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, && x \in \mathbb{R}, \\
e^{-x} &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!}\, x^n, && x \in \mathbb{R}, \\
\cosh x &= \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, && x \in \mathbb{R}, \\
\sinh x &= \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. && x \in \mathbb{R}, \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}, && x \in \mathbb{R}, \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}. && x \in \mathbb{R},
\end{aligned}$$

Example 43. Binomial series. For an real number r and any positive integer k we introduce the symbol

$$\binom{r}{k} := \frac{r(r-1)(r-2)\dots(r-k+1)}{k!}.$$

We observe that if r is a positive integer, this notation agrees with the usual binomial coefficient. Let $r \in \mathbb{R}$. Define

$$f(x) = (1+x)^r = \exp(r\log(1+x)) \quad \text{for any } x > -1.$$

Then $f'(x) = r(1+x)^{r-1}$, and by induction we verify that

$$f^{(k)}(x) = \binom{r}{k}(1+x)^{r-k}, \quad x > -1.$$

There is a so-called *Cauchy's form of remainder* in Taylor's theorem:

$$R_n(f; a, x) := \frac{f^{(n+1)}(c)}{n!}(x-a)(x-c)^n, \quad c \text{ between } a \text{ and } x.$$

This can be used to show that the Taylor series for the binomial function converges for all $x \in (-1, 1)$:

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n, \quad -1 < x < 1.$$

Example 44. Find Maclaurin series for $1/\sqrt{1+x}$. We have

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n \, n!} x^n$$

for all $\quad -1 < x \leq 1$.

Example 45. The Maclaurin series for $\arcsin x$ is obtained by expanding $(1-x^2)^{-1/2}$ into a binomial series and integrating:

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n}, \quad -1 < x < 1.$$

Pointwise and uniform convergence. Let (f_n) be a sequence of functions defined on the same interval I, and suppose that for each fixed $x \in I$, the real sequence $(f_n(x))$ converges to a limit which we denote by $f(x)$. We say that the sequence (f_n) converges *pointwise* to f, and the function f is called the *limit* of the sequence (f_n). Formally, for each $\varepsilon > 0$ and each $x \in I$ there exists $N = N(\varepsilon, x) > 0$ such that

$$n > N(\varepsilon, x) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

If for each $\varepsilon > 0$ there is $N(\varepsilon)$ which depends only on ε but not on $x \in I$, we say that the sequence converges *uniformly on I*.

How can we tell the difference between pointwise and uniform convergence? If the convergence is uniform, we have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in I$ if $x > N(\varepsilon)$. Taking the supremum, we get

$$\sup_{x \in I} |f_n(x) - f(x)| \leq \varepsilon \text{ if } n > N(\varepsilon). \tag{35}$$

And conversely, if for each $\varepsilon > 0$ there exist $N(\varepsilon)$ such that (35) holds, the convergence is uniform. This proves the following result.

Theorem 45. *A sequence* (f_n) *of functions converges uniformly on an interval* I *to a function* f *if and only if the sequence of constants given by*

$$\alpha_n := \sup_{x \in I} |f_n(x) - f(x)|$$

converges to 0 *as* $n \to \infty$.

The importance of uniform convergence is that it preserves a number of properties possessed by the functions in the sequence.

Theorem 46. *Suppose that a sequence* (f_n) *of functions converges uniformly on an interval* I *to a function* f. *Then the following are true:*

(i) *If each* f_n *is bounded, then so is* f.

(ii) *If each* f_n *is continuous, then so is* f.

(iii) *If* I *is bounded and each* f_n *is Riemann integrable on* I, *then so is* f.

A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with radius of convergence $R > 0$ and centre x_0 converges pointwise on the interval $(x_0 - R, x_0 + R)$ to some limit $f(x)$. We show that for any r satisfying $0 < r < R$, the power series converges uniformly on $[x_0 - r, x_0 + r]$; interestingly, the series may not converge uniformly on the interval $(x_0 - R, x_0 + R)$.

Example 46. Let us consider the power series $\sum_{n=0}^{\infty} x^n$. The centre is 0, and we know that the radius of convergence is $R = 1$; the limit function is

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1.$$

But

$$\sup_{-1<x<1} \left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| = \sup_{-1<x<1} \frac{|x|^{n+1}}{1-x} = \infty,$$

which shows that the series does not converge uniformly on $(-1, 1)$.

Restrict the interval $(-1, 1)$ to $[-r, r]$, where $0 < r < 1$. This time

$$\sup_{-r \le x \le r} \left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| = \sup_{-r \le x \le r} \frac{|x|^{n+1}}{1-x} = \frac{r^{n+1}}{1-r},$$

and $\lim_{n\to\infty} r^{n+1}/(1-r) = 0$. Hence the series converges uniformly on $[-r, r]$ for any r with $0 < r < 1$.

The uniform convergence of series of functions can be verified by the following result.

Theorem 47 (Weierstrass M-test)**.** *If a series* $\sum_{n=1}^{\infty} f_n(x)$ *of functions on an interval* I *satisfies the inequalities* $|f_n(x)| \le M_n$ *on* I *for all* n *and if the series* $\sum_{n=1}^{\infty} M_n$ *of constants converges, then the given series converges uniformly on* I.

For a general power series we have the following result.

Theorem 48. *Let $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a power series with radius of convergence $R > 0$. Then for any r with $0 < r < R$ the series converges uniformly on $[x_0 - r, x_0 + r]$.*

Proof. Without a loss of generality we may assume that $x_0 = 0$. Let $0 < r < R < \infty$. Choose c such that $r/R < c < 1$. By (31), $R = 1/\rho$, where $\rho = \limsup_{n\to\infty} |a_n|^{1/n}$. Since $\rho < c/r$, there is N such that $|a_n|^{1/n} \le c/r$, that is, $|a_n|r^n \le c^n$ for all $n > N$. The series of constants $\sum_{n=0}^{\infty} c^n$ converges since $0 < c < 1$. Hence by the Weierstrass M-test the power series converges uniformly on $[-r, r]$.

Fourier series. At the beginning of the 19th century Joseph Fourier studied series of the type

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \tag{36}$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt\, dt, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt\, dt, \tag{37}$$

and $f\colon \mathbb{R} \to \mathbb{R}$ is a periodic function with a period 2π, that is, $f(t + 2\pi) = f(t)$ for all t. He claimed that for such functions f the series (36) converges to $f(t)$. But in 1873 Paul Du Bois-Reymond constructed an example that showed this is not true even for *continuous* 2π-periodic functions. It was only in 1966 that a Swedish mathematician Lennart Carleson proved a general case of convergence of the Fourier series (36). Carleson received 2006 Abel prize for his achievements—this prize is equivalent to Nobel prize in mathematics.

The series (36) with the coefficients given by (37) is called the *Fourier series* of a 2π-periodic function f, and the coefficients are called the *Fourier coefficients.* What is the significance of the Fourier coefficients? We regard the functions continuous on the interval $[0, 2\pi]$ as vectors, and introduce an inner product by

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\, dt. \tag{38}$$

It can be checked that this definition satisfies the requisite properties of an inner product. In particular,

$$\|f\| = \sqrt{\langle f, f\rangle} = \left(\int_0^{2\pi} f^2(t)\, dt\right)^{1/2}$$

has all the properties of a norm. Our motivation is the well known equation valid in finite dimensional inner product spaces,

$$f = \sum_{k=0}^{n} \langle f, f_k\rangle f_k,$$

where the vectors $f_0, f_1, \dots, f_n$ are *orthonormal*, that is, $\langle f_k, f_m\rangle = 0$ if $k \neq m$, and $\|f_k\| = 1$ for all k. We wish to extend this to an infinite sum with orthonormal vectors $f_0, f_1, f_2, f_3, \dots$:

$$f = \sum_{n=0}^{\infty} \langle f, f_k\rangle f_k. \tag{39}$$

The sequence of functions

$$1,\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \cos 3t,\ \sin 3t,\ \dots,\ \cos kt,\ \sin kt,\ \dots \tag{40}$$

has most of the properties of an orthonormal sequence:

$$\begin{aligned}
\langle 1, \cos kt\rangle &= \int_0^{2\pi} \cos kt\, dt = 0,\\
\langle 1, \sin kt\rangle &= \int_0^{2\pi} \sin kt\, dt = 0,\\
\langle \cos mt, \sin kt\rangle &= \int_0^{2\pi} \cos mt \sin kt\, dt = 0, \quad \text{all } m, k,\\
\langle \cos mt, \cos kt\rangle &= \int_0^{2\pi} \cos mt \cos kt\, dt = 0, \quad m \neq k,\\
\langle \sin mt, \sin kt\rangle &= \int_0^{2\pi} \sin mt \sin kt\, dt = 0, \quad m \neq k,\\
\|1\| &= \left(\int_0^{2\pi} 1\, dt\right)^{1/2} = \sqrt{2\pi},\\
\|\cos kt\| &= \left(\int_0^{2\pi} \cos^2 kt\, dt\right)^{1/2} = \sqrt{\pi},\\
\|\sin kt\| &= \left(\int_0^{2\pi} \sin^2 kt\, dt\right)^{1/2} = \sqrt{\pi}.
\end{aligned}$$

If we normalize the functions in (40), they will form an orthonormal sequence:

$$f_0 = \frac{1}{\sqrt{2\pi}},\ f_1 = \frac{\cos t}{\sqrt{\pi}},\ f_2 = \frac{\sin t}{\sqrt{\pi}},\ f_3 = \frac{\cos 2t}{\sqrt{\pi}},\ f_4 = \frac{\sin 2t}{\sqrt{\pi}}, \dots \tag{41}$$

We can consider expansion of f in the series (39). For this we observe that in the term $\langle f, f_k\rangle f_k$ with $k > 0$ the factor $1/\sqrt{\pi}$ appears once in the Fourier coefficient $\langle f, f_k\rangle$, and once in f_k, so that

$$\langle f, f_k\rangle f_k = \frac{1}{\pi}\left(\int_0^{2\pi} f(t)\cos kt\, dt\right)\cos kt \text{ or } \frac{1}{\pi}\left(\int_0^{2\pi} f(t)\sin kt\, dt\right)\sin kt$$

depending on whether k is odd or even. For $k = 0$ we get

$$\langle f, f_0\rangle f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt.$$

While the continuity of a 2π-periodic function is not enough to ensure the convergence of its Fourier series, the continuity of f and f' is enough. In fact, we need only piecewise continuity of f and f'.

Definition. A function $g\colon [a,b] \to \mathbb{R}$ is *piecewise continuous* on $[a,b]$ if there exists a partition $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ of $[a,b]$ such that g is continuous on each subinterval (t_{k-1}, t_k), $k = 1, \ldots, n$, finite one-sided limits $g(t_k+)$ exist for $k = 0, 1, \ldots, n-1$, and finite one-sided limits $g(t_k-)$ exist for $k = 1, 2, \ldots, n$.

The functional values $g(t_k)$ at the points t_k in the above definition are irrelevant, the function may be even undefined at these points.

Example 47. The function $g\colon [-1,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 - 3, & -1 \le x < 0, \\ 2009, & x = 0, \\ x + 1, & 0 < x \le 1, \end{cases}$$

is piecewise continuous on $[-1,1]$.

Theorem 49 (Fourier series)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function with f and f' piecewise continuous on $[0, 2\pi]$. Then the Fourier series for f converges to*

(i) *$f(t)$ if t is a point of continuity of f,*

(ii) *$\frac{1}{2}(f(t+) + f(t-))$ if t is a point of discontinuity of f.*

Proof of this theorem is quite long and technical and will not be given here. Some of the ideas used in the proof appear in the solution to Problem 119 (available on LMS). For instance, one ingredient of the proof is the formula

$$\tfrac{1}{2} + \cos t + \cos 2t + \cdots + \cos nt = \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2} t}.$$

Example 48. Let f be a 2π-periodic function defined by $f(t) = t^2$ for $0 < t < 2\pi$. Sketch the graph of f and find the Fourier series for f.

The graph will repeat the parabola shape given for $0 < t < 2\pi$ on every interval $(2n\pi, 2(n{+}1)\pi)$, $n \in \mathbb{Z}$. We see that $f(0+) = 0$ and $f(0-) = 4\pi^2$. The function f has not been defined at the points $t = 2n\pi$, $n \in \mathbb{Z}$; they are the points of discontinuity. The function $f(t) = t^2$ and its derivative $f'(t) = 2t$ are piecewise continuous on $[0, 2\pi]$ with the only points of discontinuity being 0 and 2π. Hence Theorem 49 applies to f.

Now we calculate the Fourier coefficients for f:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} t^2 \cos kt \, dt = \frac{1}{\pi} \left[t^2 \frac{\sin kt}{k} - 2t \frac{-\cos kt}{k^2} + 2 \frac{-\sin kt}{k^3} \right]_0^{2\pi} = \frac{4}{k^2},$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} t^2 \sin kt \, dt = \frac{1}{\pi} \left[t^2 \frac{-\cos kt}{k} - 2t \frac{-\sin kt}{k^2} + 2 \frac{\cos kt}{k^3} \right]_0^{2\pi} = \frac{-4\pi}{k},$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} t^2 \, dt = \frac{4\pi^2}{3}.$$

Thus

$$t^2 = \frac{4\pi^2}{3} + \sum_{k=1}^{\infty} \left(\frac{4}{k^2} \cos kt - \frac{4\pi}{k} \sin kt \right), \quad 0 < t < 2\pi;$$

at the points $t = 0$ and $t = 2\pi$ the series converges to $\frac{1}{2}(f(t+) + f(t-)) = 2\pi^2$.

The preceding result for $t = 0$ gives

$$\frac{4\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} = \tfrac{1}{2}(f(0+) + f(0-)) = 2\pi^2,$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\pi^2 - \frac{1}{3}\pi^2 = \frac{\pi^2}{6}. \tag{42}$$

It is sometimes more convenient to consider a 2π-periodic function on the interval $(-\pi, \pi)$ instead of $(0, 2\pi)$. The Fourier coefficients are then integrals over $(-\pi, \pi)$.

A function may be periodic with a period other than 2π. If f is periodic with a period $2p$, it is enough to know the values of f on the interval $(-p, p)$ and use the equation $f(x + 2p) = f(x)$. In order to define the Fourier series for f we transform the interval $(-p, p)$ to the interval $(-\pi, \pi)$ by $t \mapsto \pi t/p$ for $t \in (-p, p)$. The Fourier coefficients are given by

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(t)\, dt, \;\; a_k = \frac{1}{p} \int_{-p}^{p} f(t) \cos \frac{k\pi t}{p}\, dt, \;\; b_k = \frac{1}{p} \int_{-p}^{p} f(t) \sin \frac{k\pi t}{p}\, dt. \tag{43}$$

Sheet 1: Number systems and induction

1. Using only algebraic properties A1–A5, prove the following.

   (i) (Uniqueness of zero and unit.) There is at most one real number α such that $a+\alpha = a$ for all $a \in \mathbb{R}$, and at most one real number β such that $a \cdot \beta = a$ for all $a \in \mathbb{R}$.

   (ii) (Uniqueness of opposite and reciprocal.) Given $a \in \mathbb{R}$, there is at most one real number x such that $a + x = 0$, and at most one real number y such that $a \cdot y = 1$.

   (iii) (Cancellation laws.) $(a + c = b + c) \Rightarrow (a = b)$; $\quad (ac = bc) \Rightarrow (a = b)$ if $c \neq 0$.

   (iv) Show that $(a')' = a$ for any $a \in \mathbb{R}$, and $(a^*)^* = a$ for any $a \in \mathbb{R} \setminus \{0\}$.

   (v) Show that $a \cdot 0 = 0$ for any $a \in \mathbb{R}$.
   Suggestion. $a \cdot 0 + a = a \cdot 0 + a \cdot 1 = a(0 + 1)$.

   (vi) Show that $ab' = (ab)' = a'b$ and $a'b' = ab$.
   Suggestion. $ab' + ab = a(b' + b)$.

2. Use the algebraic and order properties of $\mathbb{R}$ to prove that

   (i) $a \in \mathbb{R} \setminus \{0\} \Rightarrow a^2 > 0$.

   (ii) $0 < a < b \Rightarrow b^{-1} < a^{-1}$.

3. (Alternative characterization of supremum and infimum.) Let S be a nonempty subset of $\mathbb{R}$. Show that $M = \sup S$ if and only if (i) M is an upper bound for S, and (ii) for every $\varepsilon > 0$ there exists $x \in S$ such that $M - \varepsilon < x \leq M$. Formulate and prove an analogous characterization of $\inf S$.

4. (Archimedean property of $\mathbb{N}$.) Given $x, y \in \mathbb{R}^+$, show that there exists $n \in \mathbb{N}$ such that $ny > x$. (This is equivalent to the unboundedness of $\mathbb{N}$ in $\mathbb{R}$.)

5. If S is a subset of $\mathbb{R}$ and $c \in \mathbb{R}$, we define $c + S = \{c + x : x \in S\}$ and $cS = \{cx : x \in S\}$. If S is bounded, show that $c + S$ and cS are bounded and

$$\sup(c + S) = c + \sup S, \quad \inf(c + S) = c + \inf S,$$

$$\sup(cS) = c \sup S, \quad \inf(cS) = c \inf S \quad \text{provided } c \geq 0,$$

$$\sup(cS) = c \inf S, \quad \inf(cS) = c \sup S \quad \text{provided } c \leq 0.$$

6. Prove that the following numbers are i

$$\text{(i) } \sqrt{3}, \quad \text{(ii) } \sqrt{15}, \quad \text{(iii) } \sqrt[3]{2}, \quad \text{(iv) } \sqrt[4]{11}, \quad \text{(v) } \sqrt[5]{16}, \quad \text{(vi) } \sqrt{2} + \sqrt{3}.$$

7. If $a, b \in \mathbb{R}$ with $a < b$, show that there are infinitely many rational numbers between a and b as well as infinitely many irrational numbers.

Suggestion. Use decimal expansions to show that there is at least one $r \in \mathbb{Q}$, $a < r < b$. Then proceed by induction: If $r_n \in \mathbb{Q}$ has been found between a and b, there is r_{n+1} between a and r_n.

8. Use induction to prove the following statements for all $n \in \mathbb{N}$:

   (i) $\displaystyle\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1)$

   (ii) $\displaystyle\sum_{k=1}^{n}(2k-1) = n^2$

   (iii) $\displaystyle\sum_{k=1}^{n}(3k-2) = \tfrac{1}{2}n(3n-1)$

   (iv) $\displaystyle\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1)$

   (v) $\displaystyle\sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^2(n+1)^2 = \left(\sum_{k=1}^{n} k\right)^2$

   (vi) $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

   (vii) Define a sequence (a_n) inductively by setting $a_1 = 0$, $a_{2k} = \frac{1}{2}a_{2k-1}$, $a_{2k+1} = \frac{1}{2} + a_{2k}$. Then $a_{2k} = \frac{1}{2} - (\frac{1}{2})^k$.

9. Use induction to prove the following statements for all $n \in \mathbb{N}$:

   (i) 3 is a factor of $n^3 - n + 3$ $\quad [n^3 - n + 3 = 3 \cdot A_n$ for some $A_n \in \mathbb{N}]$

   (ii) 9 is a factor of $10^{n+1} + 3 \cdot 10^n + 5$

   (iii) 4 is a factor of $5^n - 1$

   (iv) $x - y$ is a factor of $x^n - y^n$

10. In each case find $n_0 \in \mathbb{N}$ such that the inequality holds for n_0 and then for all $n > n_0$.

    (i) $n < 2^n$, (ii) $n! > 2^n$, (iii) $2^n > 2n^3$

11. Given that $(d/dx)\log(1+x) = (1+x)^{-1}$ for all $x > -1$, derive a formula for

$$\frac{d^n}{dx^n}\log(1+x), \quad x > -1,$$

    and prove it by induction for all $n \in \mathbb{N}$.

12. Show that the square $U = (0,1) \times (0,1)$ in the real plane is equipotent to the interval $(0,1)$.

    Suggestion. Consider a correspondence based on decimal expansions sending each point (x,y) in U to a point $z \in (0,1)$.

Sheet 2: Inequalities and sequences

13. Prove that for any $a, b, c \in \mathbb{R}$,
$$|-a| = |a|, \quad |a \pm b| \geq |\,|a| - |b|\,|, \quad |a+b+c| \geq |a| - |b| - |c|.$$

14. Prove by induction that for any real numbers $a_1, \ldots, a_n$,
$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|, \quad \left|\sum_{k=1}^{n} a_k\right| \geq |a_p| - \sum_{k=1,\, k\neq p}^{n} |a_k|.$$
(Observe that in the second inequality there is only one plus sign.)

15. Let $a, b \in \mathbb{R}$ and let $0 < \varepsilon < |b|$. Prove that
$$\left|\frac{a+\varepsilon}{b+\varepsilon}\right| \leq \frac{|a|+\varepsilon}{|b|-\varepsilon}.$$

16. (Bernoulli's inequality.) Given $a > -1$, prove by induction that
$$(1+a)^n \geq 1 + na \quad \text{for all } n \in \mathbb{N}.$$

17. Let I be an interval and f a function $f\colon I \to \mathbb{R}$; f is called *convex* if
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad x, y \in \mathbb{R},\ 0 \leq t \leq 1.$$
Use induction to prove *Jensen's inequality* for a convex function f: If $x_k \in \mathbb{R}$, $t_k \in [0,1]$ for $k = 1, \ldots, n$ and $\sum_{k=1}^{n} t_k = 1$, then
$$f\left(\sum_{k=1}^{n} t_k x_k\right) \leq \sum_{k=1}^{n} t_k f(x_k).$$

18. (AM-GM inequality.) If $a_1, \ldots, a_n$ and $p_1, \ldots, p_n$ are positive real numbers with $p_1 + \cdots + p_n = 1$, use Jensen's inequality to prove the weighted AM-GM inequality and its important special case $p_k = 1/n$ for all k:
$$\sum_{k=1}^{n} p_k a_k \geq \prod_{k=1}^{n} a_k^{p_k}; \qquad \frac{1}{n}\sum_{k=1}^{n} a_k \geq \sqrt[n]{a_1 \cdots a_n} \quad \text{(AM-GM)}.$$
Write down explicitly the cases $n = 2$ and $n = 3$.
Suggestion. The function $f(x) = -\log x$ is convex.

19. Give the solutions to the following inequalities in terms of intervals. (For example, $|x| > 3$ is written as $x \in (-\infty, -3) \cup (3, \infty)$.)
(i) $|1 + 2x| \leq 4$, (ii) $|x+2| \geq 5$, (iii) $|x-5| < |x+1|$
(iv) $|x-2| < 3$ or $|x+1| < 1$, (v) $|x-2| < 3$ and $|x+1| < 1$.

20. Prove that for all $x > 0$,
$$\log x \geq \frac{x-1}{x}.$$

21. Let $a, b \geq 0$, $p > 1$ and $q = p/(p-1)$. Prove that $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$. Write down the special case for $p = 2$.

Suggestion. AM-GM inequality. Alternatively, treat one of the numbers a, b as a real variable, and define $f(x) = x^p/p + b^q/q - bx$, $x \geq 0$.

22. Let $a, b \geq 0$ and $0 < p < 1$. Prove that $(a+b)^p \leq a^p + b^p$.

Suggestion. Set $f(x) = x^p + b^p - (x+b)^p$.

23. Prove that $1 + x \leq \exp x$ for all $x \in \mathbb{R}$.

24. For any $x, y \in \mathbb{R}$ prove that $\dfrac{|x+y|}{1+|x+y|} \leq \dfrac{|x|}{1+|x|} + \dfrac{|y|}{1+|y|}$.

Suggestion. Show first the $f(u) = u/(1+u)$ is increasing for $u \geq 0$.

25. Decide which of the following sequences converge or diverge, and find the limits for the convergent sequences (ε-$N(\varepsilon)$ not required).

(i) $\dfrac{n}{2n+1}$ (ii) $\sqrt{n}$ (iii) $\dfrac{1}{\sqrt{n}}$ (iv) $\sqrt{n+1} - \sqrt{n}$ (v) $\sqrt{n}(\sqrt{n+1} - \sqrt{n})$

26. For the following sequences find the limits first using the limit theorems. Then verify the result using Cauchy's ε-$N(\varepsilon)$ definition of the limit.

(i) $\dfrac{n}{n^2+1}$ (ii) $\dfrac{2n}{n+1}$ (iii) $\dfrac{3n+1}{2n+5}$ (iv) $\dfrac{n^2-1}{2n^2+3}$

27. Show that the following sequence is increasing and bounded above by 3:

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Suggestion. AM-GM inequality for $a_1 = \cdots = a_n = 1 + n^{-1}$ and $a_{n+1} = 1$.

28. Let $0 < a < 1$. Show that the sequence $x_n = a^n$ is decreasing and bounded, and therefore convergent with the limit 0. Then use the limit translation $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} x_n$. Extend the result for $|a| < 1$ (the sandwich rule).

29. If $a > 0$, prove that $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$. (Monotonic sequence theorem.)

Suggestion. If $a > 1$, then $1 < a < a^{1+1/n}$. If $0 < a < 1$ set $b = a^{-1} > 1$.

30. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. (Monotonic sequence theorem.)

Suggestion. From the inequality $(1 + n^{-1})^n < n$ valid for $n \geq 3$ (Problem 27) deduce $(n+1)^{1/(n+1)} < n^{1/n}$ if $n \geq 3$.

31. (Harder.) Let $a > 0$ and let (x_n) be defined by choosing arbitrary $x_1 > 0$ and then $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$ for $n = 1, 2, \ldots$ Prove that (x_n) is decreasing and bounded below. Show that $x_n \to \sqrt{a}$. (This method of calculation of square roots was known in Mesopotamia 1500 BCE.)

Sheet 3: Sequences continued

32. Prove that a real sequence can have at most one limit.

33. Prove that every convergent sequence is Cauchy.

34. Prove that every Cauchy sequence which has a convergent subsequence is itself convergent.

35. Prove that every Cauchy sequence is bounded. Conclude that every convergent sequence is bounded.

36. A sequence (a_n) is called *contractive* if there exists a *contractive constant* α, $0 < \alpha < 1$, such that $|a_{n+1} - a_n| \le \alpha|a_n - a_{n-1}|$ for $n = 2, 3, \dots$ Prove that a contractive sequence is Cauchy, and therefore convergent.

    Suggestion. Emulate the argument from the Example $a_1 = 1$, $a_{n+1} = 1 + 1/a_n$ presented in this Workbook to show that for $m > n$ the following inequality holds: $|a_m - a_n| \le \alpha^{n-1}(1-\alpha)^{-1}|a_2 - a_1|$.

37. If (a_n) is a contractive sequence with the contractive constant α and the limit a, show that $|a_n - a| \le \alpha^{n-1}(1-\alpha)^{-1}|a_2 - a_1|$, $n = 1, 2, \dots$

38. Let $x_1 = 1$ and $x_{n+1} = (2 + x_n)^{-1}$ for $n = 1, 2, \dots$ Show that the sequence is contractive and find the limit.

39. We are told that the equation $x^3 - 7x + 2 = 0$ has a solution between 0 and 1. Rewrite the equation in the form $x = (x^3 + 2)/7$ and use the contractive sequence

$$x_{n+1} = \tfrac{1}{7}(x_n^3 + 2), \quad n = 1, 2, \dots$$

    to approximate the solution within 10^{-3}.

40. Let $a_1 = \alpha$ and $a_{n+1} = \sqrt{\beta + a_n}$, $n \in \mathbb{N}$, where α, β are positive real numbers. Prove that (a_n) is convergent and find the limit.

    Suggestion. Consider the cases $\sqrt{\alpha + \beta} \le \alpha$ and $\sqrt{\alpha + \beta} > \alpha$.

41. Repeat the preceding problem when $a_1 = \alpha$ and $a_{n+1} = \beta + \sqrt{a_n}$, $n \in \mathbb{N}$.

42. Find the upper and lower limits for the following sequences:

    (i) $(-1)^n(1 + n^{-1})$, (ii) $a_1 = 0$, $a_{2k} = \frac{1}{2}a_{2k-1}$, and $a_{2k+1} = \frac{1}{2} + a_{2k}$, $k \in \mathbb{N}$.

    Suggestion for (ii): $a_{2k} = \frac{1}{2} - (\frac{1}{2})^k$.

43. Let (a_n) be a bounded sequence. Show that $\liminf_n a_n \le \limsup_n a_n$. Further show that (a_n) converges if and only if $\limsup_n a_n \le \liminf_n a_n$. In this case show $\liminf_n a_n = \lim_n a_n = \limsup_n a_n$.

Sheet 4: Limits and continuity of functions

44. For each of the following rules find a suitable domain and codomain to make it bijective and find the inverse function.

    (i) $f(x) = \exp x$, (ii) $g(x) = \sin x$, (iii) $h(t) = t/(1+|t|)$, (iv) $k(s) = 2s/(1+s^2)$

45. Repeat the preceding problem for the following rules.

    (i) $9x + 2$, (ii) $x^3 + 1$, (iii) $\sqrt{2x+1}$, (iv) $1/(x-1)$
    (v) $6 - x^2$, (vi) $(x^3 + 8)^5$, (vi) $\sqrt{1 - 4x^2}$, (vii) $x^{1/3} + 2$

46. Guess the limit and then use the ε-δ definition to prove the guess.

    (i) $\lim_{x\to 4}(\frac{1}{2}x - 3)$ (ii) $\lim_{x\to 0}\dfrac{1}{1+x}$ (iii) $\lim_{x\to 4}\dfrac{1}{1+x^2}$
    (iv) $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$ (v) $\lim_{x\to 9}\dfrac{x+1}{x^2+1}$ (vi) $\lim_{x\to\infty} x^{-1}\sin x$
    (vii) $\lim_{x\to 2}\dfrac{2x^2+3x-8}{x^3-2x^2+x-12}$ (viii) $\lim_{x\to\infty}\dfrac{\log x + 2x}{3x-5}$

47. Use the sandwich rule to evaluate:

    (i) $\lim_{x\to 0} x\cos\dfrac{1}{x^2}$ (ii) $\lim_{x\to 0}\left(\sqrt{5+x^{-2}} - \sqrt{x^{-2}-1}\right)$

    Suggestion. $\sqrt{a} - \sqrt{b} = (a-b)/(\sqrt{a}. + \sqrt{b})$.

48. Write down and prove the limit and continuity theorems for functions using Heine's definition.

49. Use limit theorems and other methods to find the following limits whenever they exist:

    (i) $\lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ (ii) $\lim_{x\to\infty}\dfrac{x^4+x}{x^4+1}$ (iii) $\lim_{x\to\infty}\dfrac{7x-1}{x^2}$
    (iv) $\lim_{x\to 0+}\dfrac{\sqrt{x}}{\sqrt{7+\sqrt{x+5}}}$ (v) $\lim_{x\to 1}\dfrac{|x-1|+1}{x+|x+1|}$ (vi) $\lim_{x\to\infty}\dfrac{3x^2+1}{2x+1}$

50. Let $f\colon \mathbb{R}\to\mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x = 0$, show that it is continuous at every $a \in \mathbb{R}$. (Example: $f(x) = \exp x$.)

51. Let $f\colon (0,\infty)\to\mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y > 0$. If f is continuous at $x = 1$, show that it is continuous at every $a \in \mathbb{R}$. (Example: $f(x) = \log x$.)

52. Let $a \in \mathbb{R}$ and let $f\colon \mathbb{R}\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} ax & \text{if } x \le 0, \\ \sqrt{x} & \text{if } x > 0. \end{cases}$$

    Show that f is continuous on $\mathbb{R}$.

53. (Thomae's function.) Prove that Thomae's function defined in Example 15 is continuous at every irrational $x \in [0,1]$, and discontinuous at every rational $r \in [0,1]$.

54. Let $f\colon I \to \mathbb{R}$ be continuous on the interval I. Show that the function $|f|$ defined by $|f|(x) = |f(x)|$ is also continuous on I. Give an example of a discontinuous function f with $|f|$ continuous.

55. Let $f, g\colon I \to \mathbb{R}$ be continuous on the interval I. Show that the functions $\max(f,g)$ and $\min(f,g)$ are continuous on I.

    Suggestion: $\max(f,g) = \frac{1}{2}(f+g+|f-g|)$ and $\min(f,g) = \frac{1}{2}(f+g-|f-g|)$.

56. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x/(1+|x|)$. Show that f is continuous on $\mathbb{R}$, and that

$$\sup\{f(x) : x \in \mathbb{R}\} = 1, \qquad \inf\{f(x) : x \in \mathbb{R}\} = -1,$$

    but there are no points $x, y \in \mathbb{R}$ with $f(x) = 1$, $f(y) = -1$. (Sketch the graph.)

57. Analyze the following functions for uniform continuity on the given set S:

    (i) $f(x) = x$, $S = \mathbb{R}$ (ii) $f(x) = 1/x$, $S_1 = (0,1)$; $S_2 = (10^{-4}, 1)$
    (iii) $f(x) = x^2$, $S = (0,1)$ (iv) $f(x) = \sqrt{1-x^2}$, $S = [-1,1]$

    Suggestion. To show non-uniform continuity, find sequences (x_n), (y_n) in the set S such that $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)|$ does not converge to 0.

58. Show that $f(x) = \log x$ is uniformly continuous on $(1,\infty)$, but not on $(0,\infty)$.

    Suggestion. The mean value theorem.

59. Show that the function f defined in Problem 56 is uniformly continuous on $\mathbb{R}$.

60. Show that the cubic $x^3 - 6x + 3$ has exactly three real roots.

    Suggestion. Find $f(-3)$, $f(0)$, $f(1)$ and use the IVP.

61. Let I be an interval and let $f\colon I \to \mathbb{R}$ be continuous. Prove that $f(I)$ is an interval.

    Suggestion. An interval J is characterized by the following property: If $a, b \in J$ and $a < x < b$, then $x \in J$.

62. Let I be an interval and let $f\colon I \to f(I)$ be strictly monotonic and continuous on I. Prove that f has the inverse function $g\colon f(I) \to I$, and that g is strictly monotonic and continuous on $f(I)$. (Note that $J = f(I)$ is an interval by the preceding problem.)

Sheet 5: Differentiability

63. If $f\colon (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$, show that

$$\lim_{h\to 0+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true?

64. Let $f\colon (0,\infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at $x = 1$, show that f is differentiable at every $c \in (0,\infty)$ and $f'(c) = f'(1)/c$. Show that f is in fact infinitely differentiable.

65. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at $x = 0$, show that f is differentiable at every $c \in (0,\infty)$ and $f'(c) = f'(0)f(c)$. Show that f is in fact infinitely differentiable.

66. For the following functions decide whether they are continuous and/or differentiable at $x = 0$.

$$(1)\ f(x) = \begin{cases} -x^2, & x \le 0 \\ x, & x > 0. \end{cases} \qquad g(x) = \begin{cases} -x^2, & x \le 0 \\ x^3, & x > 0. \end{cases}$$

(Sketch graphs.)

67. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x < 0, \\ 1 + x^2, & x \ge 0. \end{cases}$$

Is f continuous at $x = 0$? Is it differentiable at $x = 0$?

68. Let $f\colon [a,b) \to \mathbb{R}$ be differentiable on (a,b), continuous on $[a,b)$, and let the limit

$$\lim_{x\to a+} f'(x) = L$$

exist. Prove that the right derivative $f'_+(a)$ exists and that $f'_+(a) = L$. Formulate for the left derivative the and prove. (*Suggestion.* MVT on $[a,x]$.)

69. (Straddle Lemma.) Let $f\colon I \to \mathbb{R}$ be differentiable at $c \in I$. Given $\varepsilon > 0$ show that there exists $\delta = \delta(\varepsilon) > 0$ such that if u, v satisfy $c-\delta < u \le c \le v < c+\delta$, then we have

$$|f(v) - f(u) - (v-u)f'(c)| \le \varepsilon |v-u|.$$

Suggestion. Subtract and add the term $f(c) - cf'(c)$ on the left side and use the triangle inequality. Why the name?

70. Use the rule for the derivative of the inverse function to prove that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0.$$

71. Repeat the preceding problem to prove

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

72. Using the MVT prove the following inequalities:

(i) $|sinx - \sin y| \leq |x-y|$ for all $x, y \in \mathbb{R}$

(ii) $|\log y - \log x| \leq \frac{1}{2}|x-y|$ for all $x, y \in [2, \infty)$

(iii) $|\sqrt[5]{x+1} - \sqrt[5]{x}| \leq (5x^{4/5})^{-1}$ for all $x > 0$

73. Use the MVT to show that if a function $f\colon (a,b) \to \mathbb{R}$ is differentiable with $f'(x) > 0$ for all x, then f is strictly iFmean va ncreasing in (a,b).

74. Use the MVT to show that if a function $f\colon (a,b) \to \mathbb{R}$ is twice differentiable in (a,b) with $f''(x) > 0$, then f is strictly convex in (a,b). (f is *strictly convex* in (a,b) if $f(tx+(1-t)y) < tf(x)+(1-t)f(y)$ for all $x, y \in (a,b)$ and $0 < t < 1$.)

75. Use l'Hôpital's rule or other methods to find the following limits:

(i) $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$ (ii) $\lim_{x\to\infty} \frac{\log x}{x}$ (iii) $\lim_{x\to 0+} \sqrt{x}\log x$

(iv) $\lim_{x\to 0+} \frac{\sqrt{x}}{\log x}$ (v) $\lim_{x\to 0} \frac{\sin x}{x}$ (vi) $\lim_{x\to 0} \left(\frac{1}{\arcsin x} - \frac{1}{\sin x}\right)$

76. Rewrite the limits (i)–(v) of the preceding problem in terms of the O and o orders of magnitudes: For instance, (v) $\lim_{x\to 0}(\sin x)/x = 1$ gives $x = O(\sin x)$ and $\sin x = O(x)$ as $x \to 0$, that is, $\sin x \asymp x$ as $x \to 0$.

77. As $x \to 0$, prove that $\cos x \asymp 1$, $\cos x \asymp 1 + x^2$, $\cot x \asymp 1/x$, $\arcsin x \asymp x$, $\arccos x \asymp 1$, $\exp x - 1 \asymp x$, $\log(1+x) \asymp x$; $x \asymp 2x$,

78. Let $f(x) = \frac{1}{2}\tan x$ for $x \in (0, \frac{1}{2}\pi)$. Estimate numerically the solution to $x = f(x)$ with $x \in (0, \frac{1}{2}\pi)$ (i) using Picard's iterations, (ii) using Newton's method ($F(x) = x - f(x)$).

79. Show that the equation $g(x) = x^3 + x - 1 = 0$ has a solution between 0 and 1. Transform the equation to the form $x = f(x)$ for a suitable contraction f which maps $[0,1]$ into itself. Use Picard's iterations to find this solution a correct to 3 decimals.

Suggestion. Test $g(0)$ and $g(1)$. Try $f(x) = 1/(x^2+1)$; use the estimate from Problem 37 with $\varepsilon = 10^{-4}$.

80. Show that the equation $g(x) = x^4 - 4x^2 - x + 4 = 0$ has a solution between $\sqrt{3}$ and 2. Transform the equation into the form $x = f(x)$ for a suitable contraction f mapping $[\sqrt{3}, 2]$ into itself. Use Picard's iterations to find this solution correct to 4 decimals.

Suggestion. Try $f(x) = \sqrt{2+\sqrt{x}}$.

Sheet 6: Riemann integral

81. Let $0 \le a < b$, let $f\colon [a,b] \to \mathbb{R}$, and let $f(x) = x^2$. If $\mathcal{P} = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ is a partition of $[a,b]$, and

$$w_k = \sqrt{\tfrac{1}{3}(x_k^2 + x_k x_{k-1} + x_{k-1}^2)}, \quad k = 1, 2, \ldots, n,$$

show that w_k can be chosen as a tag for $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, with

$$S(f; \overline{\mathcal{P}}) = \tfrac{1}{3}(b^3 - a^3).$$

Hence prove that $\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$.

82. Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Prove the following facts:

(i) If $|f(x)| \le M$ for all x, show that $\left|\int_a^b f\right| \le M(b-a)$.

(ii) If $(\overline{\mathcal{P}}_n)$ is a sequence of tagged partitions such that $\|\overline{\mathcal{P}}_n\| \to 0$, prove that

$$S(f; \overline{\mathcal{P}}_n) \to \int_a^b f(x)\,dx.$$

(iii) Suppose $g\colon [a,b] \to \mathbb{R}$ satisfies $g(x) = f(x)$ except for a finite number of points. Show that g is Rieman integrable and that the integrals of f and g are equal.

83. Construct a generalized primitive for a given function f on the interval given.

(i) $f(x) = |x|$, $x \in (-2, 1)$; (ii) $f(x) = |\sin x|$, $x \in \mathbb{R}$.

Hence find $\int_{-2}^1 |x|\,dx$ and $\int_0^{13\pi/2} |\sin x|\,dx$.

84. Let

$$g(x) = \begin{cases} x, & |x| \ge 1, \\ -x, & |x| < 1. \end{cases}$$

Show that $G(x) = \frac{1}{2}|x^2 - 1|$ is a generalized primitive for f on $[-2, 3]$, and prove that $\int_{-2}^3 g(x)\,dx = \frac{5}{2}$.

85. Sketch the graph of f on $[-1, 1]$, where

$$f(x) = \begin{cases} \sin(1/x), & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Is f Riemann integrable on $[-1, 1]$?

86. If f, g are Riemann integrable on $[a,b]$, show that so are the functions $h = \max(f,g)$ and $k = \min(f,g)$.

Suggestion. $\max(f,g) = \frac{1}{2}(f + g + |f - g|)$, $\min(f,g) = \frac{1}{2}(f + g - |f - g|)$

87. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and nonnegative on $[a,b]$ and let $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. Show that the continuity hypothesis cannot be dropped.

88. Suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable on each interval $[a,c]$, where $a \le c < b$ and bounded on the interval $[a,b]$. Prove that f is Riemann integrable on $[a,b]$ and that $\int_a^c f \to \int_a^b f$ as $c \to b$.

Suggestion. Use the sandwich theorem for integrals with functions

$$g(x) = \begin{cases} f(x), & a \le x \le c, \\ -M, & c < x \le b, \end{cases} \qquad h(x) = \begin{cases} f(x), & a \le x \le c, \\ M, & c < x \le b, \end{cases}$$

where $|f(x)| \le M$ for all $x \in [a,b]$.

89. Prove the Mean Value Theorem for integrals: If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there exists $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Suggestion. Apply the MVT to the function $F(x) = \int_a^x f(t)\,dt$ on $[a,b]$.

90. Prove a generalization of the MVT for integrals: If $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and $g(x) > 0$ for all x, show that there exists $c \in [a,b]$ such that $\int_a^b fg = f(c)\int_a^b g$. Show that this may fail if g is not positive on $[a,b]$.

Suggestion. If m, M are the minimum and maximum of f on $[a,b]$, show that $m\int_a^b g \le \int_a^b fg \le M\int_a^b g$. Apply the IVT to $m \le \int_a^b fg / \int_a^b g \le M$.

91. Prove the 'integration by parts' theorem: Let F, G be primitives to Riemann integrable functions f, g on $[a,b]$. Then

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg.$$

92. Evaluate the following integrals using the substitution theorem, carefully verifying the hypotheses.

(i) $\displaystyle\int_0^1 t\sqrt{1+t^2}\,dt$, (ii) $\displaystyle\int_0^2 t^2(1+t^3)^{-1/2}\,dt$, (iii) $\displaystyle\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt$.

93. Use substitution II to evaluate the following integrals.

(i) $\displaystyle\int_1^9 \frac{dt}{2+\sqrt{t}}$, (ii) $\displaystyle\int_1^4 \frac{\sqrt{t}\,dt}{1+\sqrt{t}}$, (iii) $\displaystyle\int_1^3 \frac{dt}{t\sqrt{t+1}}$.

Sheet 7: Numerical integration, differentiation under the integral sign, improper integrals

A calculator or a computer are suggested for the numerically more demanding problems. Some of these problems will be dealt with in a Lab Class.

94. Calculate the approximations T_4 and T_8 for the given integrals:

$$\text{(i)} \int_0^2 (1+x^2)\,dx \quad \text{(ii)} \int_0^1 e^{-x}\,dx \quad \text{(iii)} \int_0^{\pi/2} \sin x\,dx \quad \text{(iv)} \int_0^1 (1+x^2)^{-1}\,dx$$

Repeat with T_4 and T_8 replaced by S_4 and S_8.

95. Obtain bounds for the errors in Problem 94 (ii) and (iii).

96. For the integral in Problem 94 (iv) show that $|f''(x)| \le 2$ for $x \in [0,1]$ and that $|T_4(f) - \frac{1}{4}\pi| \le 1/96 < 0.0105$.

97. Use the trapezoidal rule with $n = 4$ to evaluate $\log 2 = \int_1^2 x^{-1}\,dx$. Show that $0.6866 \le \log 2 \le 0.6958$.

98. Use Simpson's rule with $n = 4$ to evaluate $\log 2$. Show that $0.6927 \le \log 2 \le 0.6933$.

99. Find the approximations T_8, T_{16}, S_8 and S_{16} for the integrals

$$\text{(i)} \int_0^1 e^{-x^2}\,dx \qquad \text{(ii)} \int_0^{\pi/2} \frac{\sin x}{x}\,dx$$

100. (Optional.) Derive the so-called *Midpoint rule* for approximate integration. The rule $M_n(f)$ is based on the partition of the interval $[a,b]$ into n equal subintervals and choosing the tags in the Riemann sums as the midpoints of the n subintervals. Show that the error estimate is given by

$$\left| \int_a^b f(x)\,dx - M_n(f) \right| \le \frac{(b-a)^3}{24n^2} M,$$

where M is an upper bound for $|f''(x)|$ on $[a,b]$.

101. (Optional.) Approximate the integrals in Problem 94 using M_4 and M_8 and compare with the results obtained using T_4 and T_8.

102. Compute $\dfrac{d}{dt}\displaystyle\int_1^2 \frac{\sin x^2 t}{x}\,dx$. Justify.

103. Compute $\dfrac{d}{dt}\displaystyle\int_1^2 \frac{e^{-x^2 t}}{x}\,dx$ at $t = 1$. Justify.

104. Differentiating $F(t) := \int_0^1 \frac{dx}{1+tx}$ $(t > -1)$ under the integral sign calculate the integrals

$$\int_0^1 \frac{x^n\,dx}{(1+x)^{n+1}} \text{ for } n = 1 \text{ and } n = 2.$$

105. Given that $\int_0^\pi \frac{dx}{t - \cos x} = \frac{\pi}{\sqrt{t^2-1}}$ for $t > 1$, find $\int_0^\pi \frac{dx}{(2-\cos x)^2}$.

106. Starting from $F(t) := \int_0^1 x^t\,dx = (t+1)^{-1}$ for $t > 0$, by repeated differentiation of F under the integral sign show that

$$\int_0^1 x^t \log^n x\,dx = \frac{(-1)^n\,n!}{(t+1)^{n+1}}, \quad t > 0,\ n = 0, 1, 2, \dots$$

Suggestion. Consider t in an interval $[t_1, t_2]$, where $t_1 > 0$, and verify that $(\partial^n/\partial t^n)\,x^t = x^t \log^n x$ is continuous on $[0,1] \times [t_1, t_2]$ after an appropriate extension at the points $(t, 0)$. (Note that $x^t = \exp(t \log x)$.)

107. Decide whether the following integrals are improper, and if so, explain why. Then evaluate them or show that they diverge:

(i) $\int_0^\infty e^{-2x}\,dx$ (ii) $\int_{-1}^1 \frac{dx}{(x+1)^{2/3}}$ (iii) $\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}}$

(iv) $\int_0^\infty xe^{-x}\,dx$ (v) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ (vi) $\int_0^\infty \frac{x\,dx}{(1+2x^2)^{2/3}}$

108. Determine whether the given improper integrals converge or diverge and justify your claim.

(i) $\int_0^\infty \frac{dx}{1+\sqrt{x}}$; (ii) $\int_0^\infty \exp(-x^3)\,dx$; (iii) $\int_0^\infty \frac{e^x\,dx}{x+1}$; (iv) $\int_e^\infty \frac{dx}{x \log^p x}$, $p > 0$

109. Determine whether the improper integrals $\int_0^1 \frac{dx}{\sqrt{x+x^2}}$ and $\int_1^\infty \frac{dx}{\sqrt{x+x^2}}$ converge or diverge.

Suggestion. Check that $(x+x^2)^{-1/2} \asymp 1/\sqrt{x}$ as $x \to 0+$ and $(x+x^2)^{-1/2} \asymp 1/x$ as $x \to \infty$.

110. Discuss the convergence of the following integrals.

(i) $\int_0^\infty \frac{dx}{x^{2/3}(x^2+1)^{1/4}}$; (ii) $\int_1^\infty \frac{(x^2+1)^{1/8}dx}{x^{3/2}}$; (iii) $\int_0^\infty \frac{dx}{x^{1/2}(x^2+1)^{1/4}}$.

Answers:

102. $(1/2t)(\sin 4t - \sin t)$ for $t \neq 0$, $\frac{3}{2}$ for $t = 0$; 103. $-(e^3-1)/(2e^4)$; 104. $\log 2 - \frac{1}{2}$; $\log 2 - \frac{5}{8}$; 105. $(2/3^{3/2})\pi$; 107. (i) $\frac{1}{2}$; (ii) $3 \cdot 2^{1/3}$; (iii) 3; (iv) 1; (v) π; (vi) $\frac{1}{2}$; 108. (i) diverges; (ii) converges; (iii) diverges; (iv) converges for $p > 1$, diverges for $0 < p \leq 1$; 109. converges; diverges; 110. (i) converges; (ii) converges; (iii) diverges.

Sheet 8: Series, Taylor polynomials

111. Use the integral test to determine if the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^5}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ (d) $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$

112. Use the comparison test to show that the following series are convergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$

113. Use the comparison test to show that the following series are divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n+1}$ (b) $\sum_{n=2}^{\infty} \frac{1}{n-1}$

114. Use the comparison test to determine if the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (c) $\sum_{n=1}^{\infty} \frac{2}{3^n+1}$ (d) $\sum_{n=1}^{\infty} \frac{1+3^n}{1+4^n}$

115. Use the ratio test to determine the convergence or divergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (c) $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ (d) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

116. Test the following alternating series for convergence or divergence.

(a) $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \cdots$ (b) $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$ (d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

117. Determine whether the following series are absolutely convergent.

(a) $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$

118. Determine whether the following series are convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ (b) $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^7}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ (e) $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ (f) $\sum_{n=1}^{\infty} \frac{\sin n}{1+n^2}$

119. * Let (a_k) be a monotonic sequence with $a_n \to 0$. Show that the series $\sum_{k=1}^{\infty} a_k \sin kx$ converges conditionally for each $x \in \mathbb{R}$.

Suggestion. Dirichlet's test with $b_k := \sin kx$. Show that $|B_n| \leq 1/\sin(x/2)$ if $\sin(x/2) \neq 0$. (Full solution available on LMS.)

Taylor polynomials

120. Write quadratic approximation for the given function near the point specified and use it to approximate the indicated value. Estimate the error and find the smallest interval you can be sure contains the value.

  (i) $f(x) = \sqrt[3]{x}$ near 8; approximate $\sqrt[3]{9}$

  (ii) $f(x) = x^{-1}$ near 1; approximate $1/1.02$

  (iii) $f(x) = e^x$ near 0; approximate $e^{-0.5}$

121. (a) From Taylor's theorem, write down an expression for the remainder $R_n(x)$ when the Taylor polynomial of degree n for e^x (about $x = 0$), $P_n(x)$, is subtracted from e^x. In what interval does the unknown constant c lie, if $x > 0$?

(b) Show that the remainder has the bounds, if $x > 0$,

$$\frac{x^{n+1}}{(n+1)!} < R_n(x) < e^x \frac{x^{n+1}}{(n+1)!}$$

and use the sandwich rule to show that $R_n(x) \to 0$ as $n \to \infty$. This proves that the Taylor series for e^x does converge to e^x, for any $x > 0$.

122. (a) Find the Taylor polynomial of degree 9 centred at $x = 0$ for the function $f(x) = \sinh x$.

(b) From Taylor's theorem, write down an expression for the remainder $R_9(x)$ and hence bound the error in approximating $\sinh 1$ by using the Taylor polynomial from part (a).

(c) Explain why, for this example, you can get a tighter bound by using a bound for $R_{10}(x)$. Find this bound.

You may use the facts: $\sinh 1 < \cosh 1 < 3$; $10! \approx 3.6 \times 10^6$.

123. The $\sqrt{x}$ button on your calculator has just broken. To overcome this disaster, do the following in order to approximate $\sqrt{6}$.

(a) By writing $\sqrt{6} = 2\left(1 + \frac{1}{2}\right)^{1/2}$ use a degree 5 Taylor polynomial (about $x = 0$) for $(1 + x)^{1/2}$ to obtain an approximate value for $\sqrt{6}$. Use a calculator or spreadsheet to calculate this value to 6 decimal places.

(b) Use Taylor's theorem to write down an expression for the error $R_5(x)$, where

$$(1 + x)^{1/2} = P_5(x) + R_5(x)$$

In what interval does the unknown constant c lie?

(c) We want to estimate the error made in approximating $\sqrt{6}$, as in part (a). Find an upper bound for $|R_5|$ and use this to bound the error in your estimate from part (a). Use a calculator or spreadsheet to calculate this value to 6 decimal places.

(d) Check that the Taylor polynomial of degree 5 gives an actual error smaller than the bound you derived in part (c). Use the 'exact' value of $\sqrt{6}$ as given by a calculator.

(e) Why would the alternative expression $\sqrt{6} = (1+5)^{1/2}$ not be a sensible way to proceed?

Answers:

111. (a) convergent (b) convergent (c) divergent (d) convergent
112. (a) convergent (b) convergent
113. (a) divergent (b) divergent
114. (a) divergent (b) divergent (c) convergent (d) convergent
115. (a) convergent (b) convergent (c) divergent (d) convergent
116. (a) convergent (b) divergent (c) convergent (d) convergent
117. (a) Yes (b) No (c) Yes (d) No
118. (a) convergent, by alternating series test; (b) divergent, by divergence test; (c) convergent, by integral test; (d) divergent by comparison test; (e) convergent by ratio test; (f) convergent by absolute convergence and comparion test
120. (i) $\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$, $\sqrt[3]{9} \approx 2.07986$,

$0 < \text{error} \le 5/(81 \times 256)$, $2.07986 < \sqrt[3]{9} < 2.08010$

(ii) $x^{-1} \approx 1 - (x-1) + (x-1)^2$, $(1/1.02) \approx 0.9804$

$-(0.02)^3 \le \text{error} < 0$, $0.980392 \le (1/1.02) < 0.9804$

(iii) $e^x \approx 1 + x + \frac{1}{2}x^2$, $e^{-0.5} \approx 0.625$

$-\frac{1}{6}(0.5)^3 \le \text{error} < 0$, $0.604 \le e^{-0.5} < 0.625$

121. (a) $R_n(x) = e^c x^{n+1}/(n+1)!$ $\quad c \in (0, x)$ since $x > 0$

(b) Since e^x is a monotonic increasing function, we can bound e^c below by $e^0 = 1$ and above by e^x.

122. (a) $P_9(x) = x + x^3/3! + x^5/5! + x^7/7! + x^9/9!$

(b) $R_9(x) = \sinh c x^{10}/10!, c \in (0, x)$ if $x > 0$; $\quad |R_9(1)| < \sinh 1/10! < \frac{1}{1.2} \times 10^{-6}$

(c) Since the Maclaurin series for $\sinh x$ has only odd powers, the Taylor polynomials of degree 9 and 10 are identical; $|R_{10}(1)| < 3/11! \approx 10^{-7}$

123. (a) Using $P_5(\frac{1}{2})$ gives the approximation $\sqrt{6} \approx 2.449951$

(b) $R_5(x) = \dfrac{\frac{1}{2}\frac{-1}{2}\cdots\frac{-9}{2}}{(1+c)^{11/2}}\,\dfrac{x^6}{6!}$; $\quad c \in (0, \frac{1}{2})$

(c) $\left|R_5\left(\frac{1}{2}\right)\right| < \dfrac{\frac{1}{2}\frac{-1}{2}\cdots\frac{-9}{2}}{1}\,\dfrac{2^{-6}}{6!}$ so error $<\approx 0.000641$

(d) Actual error ≈ 0.000461

(e) The binomial series $(1+x)^{1/2}$ has radius of convergence $R = 1$.

Sheet 9: Power series, Taylor series, Fourier series

124. Write out the first four terms of the following power series (using the convention $0! = 1$).

$$\text{(a)} \sum_{n=0}^{\infty} \frac{x^n}{n+1} \quad \text{(b)} \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} \quad \text{(d)} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+2}$$

125. Find the radius of convergence for the following power series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{x^n}{n+1} \quad \text{(b)} \sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{(n+1)^2} \quad \text{(c)} \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(d)} \sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n}}$$

126. Find the interval of convergence for the power series in the preceding problem.

127. By differentiating or integrating the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

find the sum of the following power series,

$$\text{(a)} \sum_{n=1}^{\infty} n x^{n-1} \quad (|x| < 1) \quad \text{(b)} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad (|x| < 1)$$

and hence evaluate the following series.

$$\text{(c)} \sum_{n=1}^{\infty} \frac{n}{3^{n-1}} \quad \text{(d)} \sum_{n=1}^{\infty} \frac{1}{n 2^{n+1}} \quad \text{(e)} \sum_{n=1}^{\infty} n(n-1) \left(\frac{1}{4}\right)^n$$

128. By manipulating the geometric series (that is, by replacing x, or integrating or differentiating), find the power series representation for the given function, and determine its radius of convergence R *without using the ratio test.*

$$\text{(a)} \frac{1}{1+2x} \quad \text{(b)} \frac{1}{1+x^2} \quad \text{(c)} \frac{x}{1+x} \quad \text{(d)} \frac{1}{(1+x)^2} \quad \text{(e)} \arctan x \quad \text{(f)} \log(2+x)$$

129. Find the first three nonzero terms of the Maclaurin series (Taylor series about $x = 0$) for $f(x)$. Generalise an expression for the coefficient c_n of x^n.

$$\text{(a)} f(x) = e^x \quad \text{(b)} f(x) = \sinh x \quad \text{(c)} f(x) = \frac{1}{1-x}$$

130. Find the Taylor series for $f(x)$ at the given value of a and calculate the associated radius of convergence.

$$\text{(a)} f(x) = e^x, \quad a = 2 \quad \text{(b)} f(x) = \log x, \quad a = 1 \quad \text{(c)} f(x) = \frac{1}{x^2}, \quad a = 1$$

131. Like any series, Taylor series can be added and subtracted in their (intersecting) intervals of convergence.

(a) Using the Maclaurin series for $e^x, \sinh x$ and $\cosh x$, prove the identity

$$\cosh x + \sinh x = e^x$$

(b) By defining e^{ix} as the power series obtained from substituting ix for x in the Maclaurin series for e^x, prove the identity

$$e^{ix} = \cos x + i \sin x$$

132. Express the following indefinite integrals as power series

(a) $\displaystyle\int e^{x^3}\,dx$ (b) $\displaystyle\int \frac{\sinh x}{x}\,dx$

Hence express the following definite integrals as infinite series.

(c) $\displaystyle\int_0^1 e^{x^3}\,dx$ (d) $\displaystyle\int_{-1}^1 \frac{\sinh x}{x}\,dx$

Suggestion. Use appropriate results from the preceding problem.

133. By using Maclaurin series for $\sin x, e^x$ and $\cos x$, evaluate the following limits. (Note: this is an alternative technique to l'Hôpital's rule.)

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ (c) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

Fourier series.

134. Let $f(x) = x$ for $-2 < x < 2$ be a function of period 4. Find its Fourier series on the interval $(0, 2)$.

Suggestion. The function is odd. The product $f(t)\sin(k\pi t/2)$ is even and its integral over $(-2, 2)$ is twice the integral over $(0, 2)$. Further, the product $f(t)\cos(k\pi t/2)$ is odd, and the integral over $(-2, 2)$ is equal to 0. The series will have only sine terms.

135. Let $f(x) = |x|$ for $-2 < x < 2$ be a function of period 4. Find its Fourier series on the interval $(0, 2)$. Compare with the preceding problem.

Suggestion. The function is even. The product $f(t)\sin(k\pi t/2)$ is odd and its integral over $(-2, 2)$ is 0. Further, the product $f(t)\cos(k\pi t/2)$ is even, and the integral over $(-2, 2)$ is twice the integral over $(0, 2)$. The series will have only cosine terms.

136. Let $g(x) = x^2$ for $-2 < x < 2$ be a function of period 4. Find the Fourier series for g on the interval $(0, 2)$ by integrating the Fourier series of the function in Problem 134.

Answers:

124. (a) $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3$ (b) $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$
(c) $(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\frac{1}{4}(x-1)^4$ (d) $\frac{1}{2}-\frac{1}{3}x+\frac{1}{4}x^2-\frac{1}{5}x^3$

125. (a) 1 (b) 1 (c) ∞ (d) $\frac{1}{2}$

126. (a) $[-1,1)$ (b) $[-2,0]$ (c) $(-\infty,\infty)$ (d) $[0,1)$

127. (a) $(1-x)^{-2}$ (b) $-\log(1-x)$ (c) $\frac{9}{4}$ (d) $\frac{1}{2}\log 2$ (e) $\frac{8}{27}$

128. (a) $1-2x+4x^2-8x^3+\ldots=\sum_{n=0}^{\infty}(-1)^n 2^n x^n\,;\quad R=\frac{1}{2}$
(b) $1-x^2+x^4-x^6+\ldots=\sum_{n=0}^{\infty}(-1)^n x^{2n}\,;\quad R=1$
(c) $x-x^2+x^3-x^4+\ldots=\sum_{n=0}^{\infty}(-1)^n x^{n+1}\,;\quad R=1$
(d) $1-2x+3x^2-4x^3+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}\,;\quad R=1$
(e) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\,;\quad R=1$
(f) $\log 2+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{24}-\frac{x^4}{64}+\ldots=\log 2+\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)2^{n+1}}\,;\quad R=2$

129. (a) $1+x+\frac{1}{2}x^2\,;\quad c_n=\frac{1}{n!}$ if $n\geq 0$
(b) $x+\frac{1}{6}x^3+\frac{1}{120}x^5\,;\quad n$ even: $c_n=0$, n odd: $c_n=c_{2k+1}=\dfrac{1}{(2k+1)!}$ for $k=0,1,2\ldots$
(c) $1-\frac{1}{2}x^2+\frac{1}{24}x^4\,;\quad n$ odd: $c_n=0$, n even: $c_n=c_{2k}=\dfrac{(-1)^k}{(2k)!}$ for $k=0,1,2\ldots$
(d) $x-\frac{1}{2}x^2+\frac{1}{3}x^3\,;\quad c_n=(-1)^{n-1}\frac{1}{n}$ if $n\geq 1$
(e) $1+x+x^2\,;\quad c_n=1$

130. (a) $e^2\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n!}\,;\quad R=\infty$
(b) $(x-1)-\dfrac{1}{2}(x-1)^2+\dfrac{1}{3}(x-1)^3+\cdots=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n\,;\quad R=1$
(c) $1-2(x-1)+3(x-1)^2-\cdots=\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n;\quad R=1$

132. (a) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!(3n+1)}\,x^{3n+1}+C$ (b) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)!(2n+1)}\,x^{2n+1}+C$
(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!\,(3n+1)}$ (d) $\displaystyle\sum_{n=0}^{\infty}\frac{2}{(2n+1)!\,(2n+1)}$

133. (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{24}$

134. $\frac{4}{\pi}(\sin\frac{1}{2}\pi t-\frac{1}{2}\sin\pi t+\frac{1}{3}\sin\frac{3}{2}\pi t-\cdots)$

135. $1-\frac{8}{\pi^2}(\cos\frac{1}{2}\pi t+\frac{1}{3^2}\cos\frac{3}{2}\pi t+\frac{1}{5^2}\cos\frac{5}{2}\pi t+\cdots)$

136. $C-\frac{16}{\pi^2}(\cos\frac{1}{2}\pi t-\frac{1}{2^2}\cos\pi t+\frac{1}{3^2}\cos\frac{3}{2}\pi t-\cdots)$, $C=\frac{16}{\pi^2}(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots)$