MATH 221: Calculus and Analytic Geometry
Prof. Ram, Fall 2006

HOMEWORK 13
DUE December 8, 2006

Problem A. Motion.

(1) What do distance, speed and acceleration have to do with calculus? Explain thoroughly.

(2) A particle, starting from a fixed point P, moves in a straight line. Its distance from P after t seconds is $s = 11 + 5t + t^3$ meters. Find the distance, velocity and acceleration of the particle after 4 seconds, and find the distance it travels during the 4th second.

(3) The displacement of a particle at time t is given by $x = 2t^3 - 5t^2 + 4t + 3$. Find
  (i) the time when the acceleration is 8cm/s^2, and
  (ii) the velocity and displacement at that instant.

(4) A particle moves along a straight line according to the law $s = t^3 - 6t^2 + 19t - 4$. Find
  (i) its displacement and acceleration when its velocity is 7m/s, and
  (ii) its displacement and velocity when its acceleration is 6m/s^2.

(5) A particle moves along a straight line so that after t seconds its distance from a fixed point P on the line is s meters, where $s = t^3 - 4t^2 + 3t$. Find
  (i) when the particle is at P, and
  (ii) its velocity and acceleration at these instants.

(6) A particle moves along a straight line according to the law $s = at^2 - 2bt + c$, where a, b, c are constants. Prove that the acceleration of the particle is constant.

(7) If a particle moves along a straight line so that the distance described is proportional to the square of the time elapsed prove that
  (i) the velocity is proportional to the time, and
  (ii) the rate of increase of the velocity is constant.

(8) A car starts from rest and moves a distance s meters in t seconds, where $s = a\cos t + b\sin t$. Show that the acceleration at time t is the negative of the distance traveled in t seconds.

(9) The distance s in meters traveled by a particle in t seconds is given by $s = ae^t + be^{-t}$. Show that the acceleration of the particle at time t is equal to the distance the particle travels in t seconds.

(10) A particle moves in a line according to the law $s = at^2 + bt + c$ where a, b, c are constants and s is the distance of the particle from a fixed point P after t seconds. Initially the particle is 10 cm away from P and its initial velocity is 12 cm/s. If the particle moves with a uniform acceleration of 4cm/s^2 find the distance travelled by it during the 7th second.

(11) The displacement of a particle moving in a straight line is $x = 2t^3 - 9t^2 + 12t + 1$ meters at time t. Find
  (i) the velocity and acceleration at $t = 1$ second,
 (ii) the time when the particle stops momentarily, and
(iii) the distance between two stops.

(12) A particle moves in a straight line according to the law $s = 6t^3 + 20t^2 + 9t$ where s is in centimeters and t is in seconds. Find the initial velocity and acceleration of the particle.

(13) A particle is moving on a line according to the law $s = \tan^{-1} t + at^2 + bt + c$ where a, b, c are constants. Given that, at $t = 1$, s is 3.5 m, the velocity is 3 m/s, and the acceleration is 1.5 m/s^2, find the values of a, b and c.

(14) The height of a stone thrown vertically upwards is given by $s = 49t - 4.9t^2$ where s is in meters and t is in seconds. Find the maximum height reached by the stone.

(15) A particle is moving in a vertical line according to the equation $x = 100t - 4.9t^2$ where x is in meters and t is in seconds. Find its velocity at $t = 1$. A what time is its velocity 0? What is the maximum value of s?

(16) An arrow shot vertically upwards moves according to the formula $s = 49t - 4.9t^2$ where s is in meters and t is in seconds. Find the time that it takes to reach a height of 117.6 meters. What is its velocity after 8 seconds? How long before it hits the ground?

(17) A ball projected vertically upwards has equation of motion $s = ut - 4.9t^2$ where s is in meters and t is in seconds and u is the initial velocity. If the maximum height reached by the ball is 44.1 meters find the value of u.

(18) A shot fired vertically upwards is known to be at a point A at the end of 2 seconds and also again there after 3 more seconds. The equation of motion of the bullet is $s = ut - 4.9t^2$ where s is in meters, t is in seconds and u is the initial velocity of the bullet. Find the height of the point A above the point where the shot is fired.

(19) A particle falls from the top of a tower and in the last second before it hits the ground it falls 9/25 of the total height of the tower. Find the height of the tower.

Problem B. Applications of the exponential function.

(1) Find all functions $y(t)$ such that $\dfrac{dy}{dt} = ky$ and $y(a) = b$, where k, a and b are constants.

(2) What is the idea behind computing radioactive decay? Explain why exponential functions arise.

(3) What is the idea for computing population growth? Explain why exponential functions are used.

(4) Explain why exponential functions are used to compute money owed on loans. Explain why the limit $\lim\limits_{n \to \infty} \left(1 + \dfrac{a}{n}\right)^n$ is used for computing interest.

(5) What is the idea for computing temperatures of objects during cooling? Explain why exponential functions appear.

(6) If you borrow $500 on your credit card at 14% interest find the amounts due at the end of 2 years if the interest is compounded (a) annually, (b) quarterly, (c) monthly, (d) daily, (e) hourly, (f) continuously.

(7) If you buy a $24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much your monthly payments are if the interest is compounded continuously.

(8) If you buy a $24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much your payment would be if you paid it all off in one big payment at the end of 3 years.

(9) If you buy a $24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much interest you pay during the first month.

(10) If you buy a $200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much your monthly payments are if the interest is compounded continuously.

(11) If you buy a $200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much your payment would be if you paid it all off in one big payment at the end of 30 years.

(12) If you buy a $200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much interest you pay during the first month.

(13) A roast turkey is taken from an oven when its temperature has reached 185° F and is placed on a table in a room where the temperature is 75° F. Assume that it cools

at a rate proportional to the difference between its current temperature and the room temperature.

(a) If the temperature of the turkey is 150° F after half an hour, what is the temperature after 45 min?

(b) When will the turkey have cooled to 100° F?

(14) Radiocarbon dating works on the principle that ^{14}C decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ^{14}C as does plant material on earth today. Estimate the age of the parchment.

(15) After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half life of radon-222?

(b) How long would it take the sample to decay to 10% of its original amount?

(16) Polonium-210 has a half life of 140 days.

(a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.

(b) Find the mass after 100 days.

(c) When will the mass be reduced to 10 mg?

(d) Sketch the graph of the mass function.

(17) If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0, find the number at time t.

(18) If a radioactive substance disintegrates at a rate proportional to the amount present how much of the substance remains at time t if the initial amount is Q_0?

(19) If an object cools at a rate proportional to the the difference between its temperature and the temperature of its surroundings, the initial temperature of the object is T_0 and the temperature of the surroundings are a constant temperature S what is the temperature of the object at time t?

(20) Current agricultural experts believe that the worlds farms can feed about 10 billion people. The 1950 world population was 2517 million and the 1992 world population was 5.4 billion. When can we expect to run out of food?

(21) Suppose that the GNP in a country is increasing at an annual rate of 4 percent. How many years, at that rate of growth, are required to double the present GNP?

(22) What percent of a sample of $^{226}_{88}$Ra remains after 100 years? The half life of $^{226}_{88}$Ra is 1620 years.

(23) A sample contains 4.6 mg of $^{131}_{53}$I. How many mg will remain after 3.0 days? The half life of $^{131}_{53}$I is 8.0 days.

(24) The majority of naturally occuring rhenium is $^{187}_{75}$Re, which is radioactive and has a half life of 7×10^{10} years. In how many years will 5% of the earth's $^{187}_{75}$Re decompose?

(25) A piece of paper from the Dead Sea scrolls was found to have a $^{14}_{6}$C/$^{12}_{6}$C ratio 79.5% of that in a plant living today. Estimate the age of the paper, given that the half life of $^{14}_{6}$C is 5720 years.

(26) The charcoal from ashes found in a cave gave a ^{14}C activity of 8.6 counts per gram per minute. Calculate the age of the charcoal (wood from a growing tree gives a comparable count of 15.3). For ^{14}C the half life is 5720 years.

(27) In a certain activity meter, a pure sample of $^{90}_{38}$Sr has an activity (rate of decay) of 1000.0 disintegrations per minute. If the activity of this sample after 2.00 years is 953.2 disintegrations per minute, what is the half life of $^{90}_{38}$Sr.

(28) A sample of a wooden artifact from an Egyptian tomb has a ^{14}C/^{12}C ratio which is 54.2 % of that of freshly cut wood. In approximately what year was the old wood cut? The half life of ^{14}C is 5720 years.

Problem C. Logarithmic differentiation.

(1) Find $\dfrac{dy}{dx}$ when $y = \dfrac{(x+2)^{5/2}}{(x+6)^{1/2}(x+3)^{7/2}}$.

(2) Find $\dfrac{dy}{dx}$ when $y = (x+1)^2(x-2)^3(x+4)\ln x$.

(3) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\dfrac{(x-a)(x-b)}{(x-p)(x-q)}}$.

(4) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\ln x}$.

(5) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\cos x}$.

(6) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\tan x} + (\tan x)^{\sin x}$.

(7) Find $\dfrac{dy}{dx}$ when $x^y + y^x = a$.

(8) Find $\dfrac{dy}{dx}$ when $x + y = x^y$.

5

(9) Find $\dfrac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$.

(10) Find $\dfrac{dy}{dx}$ when $y = a^x + e^{\tan x} + (\cot x)^{\cos x}$.

(11) Find $\dfrac{dy}{dx}$ when $y = (\tan x)^{\cot x}$.

(12) Find $\dfrac{dy}{dx}$ when $y = x^x + x^{1/x}$.

(13) Find $\dfrac{dy}{dx}$ when $y = (\sec x)^{\csc x} + (\csc x)^{\sec x}$.

(14) Find $\dfrac{dy}{dx}$ when $y = \log_y x$.

(15) Find $\dfrac{dy}{dx}$ when $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdot^{\cdot^{\cdot}}}}}$.

(16) Find $\dfrac{dy}{dx}$ when $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.

(17) Find $\dfrac{dy}{dx}$ when $y = x^{y^x}$.

(18) Find $\dfrac{dy}{dx}$ when $y = x^{1/x}$.

(19) Find $\dfrac{dy}{dx}$ when $y = \left(\dfrac{x^x + x^{-x}}{x^x - x^{-x}} \right)^{1/2}$.

(20) If $x^m y^n = (x + y)^{m+n}$ show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Problem D. L'Hopital's rule.

(1) State L'Hôpital's rule and give an example which illustrates how it is used.

(2) Explain why L'Hôpital's rule works. Hint: Expand the numerator and the denominator in terms of Δx.

(3) Give three examples which illustrate that a limit problem that looks like it is coming out to 0/0 could be really getting closer and closer to almost anything and must be looked at a different way.

(4) Give three examples which illustrate that a limit problem that looks like it is coming out to 1^∞ could be really getting closer and closer to almost anything and must be looked at a different way.

(5) Give three examples which illustrate that a limit problem that looks like it is coming out to 0^0 could be really getting closer and closer to almost anything and must be looked at a different way.

(6) Evaluate $\lim\limits_{x\to 1} \dfrac{x^2 + 3x - 4}{x - 1}$. You may use L'Hôpital's rule.

(7) Evaluate $\lim\limits_{x\to 1} \dfrac{x^a - 1}{x^b - 1}$. You may use L'Hôpital's rule.

(8) Evaluate $\lim\limits_{x\to 1} \dfrac{\ln x}{x - 1}$. You may use L'Hôpital's rule.

(9) Evaluate $\lim\limits_{x\to \pi} \dfrac{\tan x}{x}$. You may use L'Hôpital's rule.

(10) Evaluate $\lim\limits_{x\to 3\pi/2} \dfrac{\cos x}{x - (3\pi/2)}$. You may use L'Hôpital's rule.

(11) Evaluate $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\sqrt{x}}$. You may use L'Hôpital's rule.

(12) Evaluate $\lim\limits_{x\to \infty} \dfrac{(\ln x)^3}{x^2}$. You may use L'Hôpital's rule.

(13) Evaluate $\lim\limits_{x\to 0} \dfrac{6^x - 2^x}{x}$. You may use L'Hôpital's rule.

(14) Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x - (x^2/2)}{x^3}$. You may use L'Hôpital's rule.

(15) Evaluate $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$. You may use L'Hôpital's rule.

(16) Evaluate $\lim\limits_{x\to \infty} \dfrac{\ln(1 + e^x)}{5x}$. You may use L'Hôpital's rule.

(17) Evaluate $\lim\limits_{x\to 0} \dfrac{\tan \alpha x}{x}$. You may use L'Hôpital's rule.

(18) Evaluate $\lim\limits_{x\to 0} \dfrac{2x - \sin^{-1} x}{2x + \cos^{-1} x}$. You may use L'Hôpital's rule.

(19) Evaluate $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$. You may use L'Hôpital's rule.

(20) Evaluate $\lim\limits_{x\to\infty} e^{-x}\ln x$. You may use L'Hôpital's rule.

(21) Evaluate $\lim\limits_{x\to\infty} x^3 e^{-x^2}$. You may use L'Hôpital's rule.

(22) Evaluate $\lim\limits_{x\to\pi} (x-\pi)\cot x$. You may use L'Hôpital's rule.

(23) Evaluate $\lim\limits_{x\to 0} x^{-4} - x^{-2}$. You may use L'Hôpital's rule.

(24) Evaluate $\lim\limits_{x\to 0} x^{-1} - \csc x$. You may use L'Hôpital's rule.

(25) Evaluate $\lim\limits_{x\to\infty} x - \sqrt{x^2-1}$. You may use L'Hôpital's rule.

(26) Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{x^3}{x^2-1} - \dfrac{x^3}{x^2+1}\right)$. You may use L'Hôpital's rule.

(27) Evaluate $\lim\limits_{x\to 0^+} x^{\sin x}$. You may use L'Hôpital's rule.

(28) Evaluate $\lim\limits_{x\to 0} (1-2x)^{1/x}$. You may use L'Hôpital's rule.

(29) Evaluate $\lim\limits_{x\to\infty} (1+3/x+5/x^2)^x$. You may use L'Hôpital's rule.

(30) Evaluate $\lim\limits_{x\to\infty} x^{1/x}$. You may use L'Hôpital's rule.

(31) Evaluate $\lim\limits_{x\to 0^+} (\cot x)^{\sin x}$. You may use L'Hôpital's rule.

(32) Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x$. You may use L'Hôpital's rule.

(33) Evaluate $\lim\limits_{x\to 0^+} (-\ln x)^x$. You may use L'Hôpital's rule.