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1 Definitions

The **positive integers** are $1, 2, 3, 4, 5, 6, \ldots$.

The **nonnegative integers** are $0, 1, 2, 4, 5, 6, \ldots$.

The **rational numbers** are

$$\frac{a}{b}\,, \qquad a \text{ an integer, } b \text{ an integer, } b \neq 0.$$

The **real numbers** are all possible decimal expansions.

The **complex numbers** are $a + bi$, where a and b are real numbers and i is a number such that $i^2 = -1$.

If n is a positive integer then n**-factorial** is

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

If n is a positive integer then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}.$$

This satisfies

$$a^{x+y} = a^x a^y \qquad \text{and} \qquad a^1 = a,$$

which forces

$$a^x = e^{x \ln a}, \qquad \text{where } \ln a \text{ is such that } e^{\ln a} = a.$$

A **function** is a creature that eats a number, chews on it, and spits out a new number. What a function spits out depends only on what goes in.

The **inverse function** to a function f is backwards of f. The inverse function is not usually a function because what it spits out depends on its mood. For example:

$$\sqrt{x} \text{ is the inverse function to } x^2$$

and

$$\sqrt{9} = 3 \text{ on good Mondays}, \quad \text{and} \quad \sqrt{9} = -3 \text{ on bad Tuesdays}.$$

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,
\end{aligned}$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Define

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\
\sinh x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\
\cosh x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,
\end{aligned}$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \mathrm{sech} x = \frac{1}{\cosh x}, \quad \mathrm{csch} x = \frac{1}{\sinh x}.$$

$\sqrt{x}$ is the inverse function to x^2.

$\ln x$ is the inverse function to e^x.

$\sin^{-1} x$ is the inverse function to $\sin x$.

$\cos^{-1} x$ is the inverse function to $\cos x$.

$\tan^{-1} x$ is the inverse function to $\tan x$.

$\cot^{-1} x$ is the inverse function to $\cot x$.

$\sec^{-1} x$ is the inverse function to $\sec x$.

$\csc^{-1} x$ is the inverse function to $\csc x$.

$\sinh^{-1} x$ is the inverse function to $\sinh x$.

$\cosh^{-1} x$ is the inverse function to $\cosh x$.

$\tanh^{-1} x$ is the inverse function to $\tanh x$.

$\coth^{-1} x$ is the inverse function to $\coth x$.

$\mathrm{sech}^{-1} x$ is the inverse function to $\mathrm{sech} x$.

$\mathrm{csch}^{-1} x$ is the inverse function to $\mathrm{csch} x$.

2 Numbers

At some point humankind wanted to count things and discovered the **positive integers**,

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for counting something, BUT what if you don't have anything? How do we talk about nothing, nulla, zilch? ... and so we discovered the **nonnegative integers**,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for adding,

$$5+3=8,\ \ 0+10=10,\ \ 21+37=48,$$

BUT not so great for subtraction,

$$5-3=2,\ \ 2-0=2,\ \ 12-34=???.$$

... and so we discovered the **integers**

$$\ldots,-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ .$$

GREAT for adding, subtracting and multiplying,

$$3\cdot 6=18,\ \ -3\cdot 2=-6,\ \ 0\cdot 7=0,$$

BUT not so great if you only want part of the sausage ..., ... and so we discovered the **rational numbers**,

$$\frac{a}{b}\,, \qquad a \text{ an integer, } b \text{ an integer, } b\neq 0.$$

GREAT for addition, subtraction, multiplication, and division, BUT not so great for finding $\sqrt{2}=????$, ... and so we discovered the **real numbers**,

all decimal expansions.

Examples:

$$\begin{aligned} \pi &= 3.1415926\ldots\ , \\ e &= 2.71828\ldots\ , \\ \sqrt{2} &= 1.414\ldots\ , \\ 10 &= 10.0000\ldots\ , \end{aligned} \qquad\qquad \begin{aligned} \tfrac{1}{3} &= .3333\ldots\ , \\ \tfrac{1}{8} &= .125 = .125000000\ldots\ , \end{aligned}$$

GREAT for addition, subtraction, multiplication, and division, BUT not so great for finding $\sqrt{-9}=????$, ... and so we discovered the **complex numbers**,

$$a+bi, \qquad a \text{ a real number, } b \text{ a real number, } i=\sqrt{-1}\}.$$

Examples: $3+\sqrt{2}i$, $\quad 6=6+0i$, $\quad \pi+\sqrt{7}i$, and

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i.$$

GREAT.

Addition: $(3+4i)+(7+9i) = 3+7+4i+9i = 10+13i.$

Subtraction: $(3+4i)-(7+9i) = 3-7+4i-9i = -4-5i.$

Multiplication:

$$\begin{aligned}(3+4i)(7+9i) &= 3(7+9i)+4i(7+9i)\\ &= 21+27i+28i+36i^2\\ &= 21+55i-36\\ &= -15+55i.\end{aligned}$$

Division:

$$\frac{3+4i}{7+9i} = \frac{(3+4i)}{(7+9i)}\frac{(7-9i)}{(7-9i)} = \frac{21-27i+28i+36}{49-63i+63i+81}$$

$$= \frac{57+i}{130} = \frac{57}{130} + \frac{1}{130}i.$$

Square Roots: We want $\sqrt{-3+4i}$ to be some $a+bi$.

$$\text{If} \qquad \sqrt{-3+4i} = a+bi$$

then

$$\begin{aligned}-3+4i = (a+bi)^2 &= a^2+abi+abi+b^2i^2\\ &= a^2-b^2+2abi.\end{aligned}$$

So

$$a^2-b^2=-3 \qquad \text{and} \qquad 2ab=4.$$

Solve for a and b.

$$b = \frac{4}{2a} = \frac{2}{a}. \qquad \begin{aligned}&\text{So} \quad a^2-\left(\frac{2}{a}\right)^2 = -3.\\ &\text{So} \quad a^2 - \frac{4}{a^2} = -3.\\ &\text{So} \quad a^4-4=-3a^2.\\ &\text{So} \quad a^4+3a^2-4=0.\\ &\text{So} \quad (a^2+4)(a^2-1)=0.\end{aligned}$$

So $a^2=-4$ or $a^2=1$.
So $a=\pm 1$, and $b=\frac{2}{\pm 1}=2$ or -2.
So $a+bi=1+2i$ or $a+bi=-1-2i$.
So $\sqrt{-3+4i}=\pm(1+2i)$.

Graphing:

Factoring:

$$x^2 + 5 = (x + \sqrt{5}\, i)(x - \sqrt{5}\, i),$$
$$x^2 + x + 1 = \left(x - \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)\right)\left(x - \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)\right)$$

This is REALLY why we like the complex numbers. The **fundamental theorem of algebra** says that ANY POLYNOMIAL (for example, $x^{12673} + 2563x^{159} + \pi x^{121} + \sqrt{7}\, x^{23} + 9621\frac{1}{2}$) can be factored completely as

$$(x - u_1)(x - u_2) \cdots (x - u_n)$$

where $u_1, \ldots, u_n$ are complex numbers.

3 The exponential function

$$\underset{x}{\text{input}} \longrightarrow \boxed{e^x} \longrightarrow \underset{e^x}{\text{output}}$$

The **exponential function** is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$

where k-**factorial** is

$$k! = k(k-1)(k-2) \cdots 3 \cdot 2 \cdot 1, \quad \text{for } k = 1, 2, 3, \ldots.$$

For example,

$$\begin{aligned} 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 3! &= 3 \cdot 2 \cdot 1. \end{aligned}$$

Why would anyone be so crazy as to write down such a horrible mess as e^x??

Example: Is there a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

that changes addition into multiplication??,

$$f(x) f(y) = f(x + y).$$

If so

$$\begin{aligned} f(x+y) &= c_0 + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + c_4(x+y)^4 + c_5(x+y)^5 + \cdots \\ &= c_0 \\ &\quad + c_1 x + c_1 y + \\ &\quad + c_2 x^2 + 2c_2 xy + c_2 y^2 + \\ &\quad + c_3 x^3 + 3c_3 x^2 y + 3c_3 xy^2 + c_3 y^3 + \\ &\quad + c_4 x^4 + 4c_4 x^3 y + 6c_4 x^2 y^2 + 4c_4 xy^3 + c_4 y^4 + \\ &\quad + \cdots \end{aligned}$$

must be equal to

$$\begin{aligned} f(x)f(y) &= (c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots)(c_0 + c_1y + c_2y^2 + c_3y^3 + c_4y^4 + \cdots) \\ &= c_0^2 + c_0c_1x + c_0c_2x^2 + c_0c_3x^3 + c_0c_4x^4 + \cdots \\ &\qquad + c_0c_1y + c_1^2xy + c_1c_2x^3y + c_1c_4x^4y + \cdots \\ &\qquad + \cdots . \end{aligned}$$

Comparing terms in these two expressions gives

$$\begin{array}{llllll} c_0^2 = c_0, & c_0c_1 = c_1, & c_0c_2 = c_2, & c_0c_3 = c_3, & c_0c_4 = c_4, & \ldots, \\ & c_0c_1 = c_1, & c_1^2 = 2c_2, & c_1c_2 = 3c_3, & c_1c_3 = 4c_4, & c_1c_4 = 5c_5, \quad \ldots, \end{array}$$

So

$$c_0 = 1, \quad c_2 = \frac{c_1^2}{2}, \quad c_1\frac{c_1^2}{2} = 3c_3, \quad c_1\frac{c_1^3}{3\cdot 2} = 4c_4, \quad c_1\frac{c_1^4}{4\cdot 3\cdot 2} = 5c_5, \quad \ldots .$$

So

$$c_0 = 1, \quad c_2 = \frac{c_1^2}{2}, \quad c_3 = \frac{c_1^3}{3\cdot 2\cdot 1}, \quad c_4 = \frac{c_1^4}{4\cdot 3\cdot 2\cdot 1}, \quad c_5 = \frac{c_1^5}{5\cdot 4\cdot 3\cdot 2\cdot 1}, \quad \ldots .$$

So

$$\begin{aligned} f(x) &= 1 + c_1x + \frac{c_1^2}{2}x^2 + \frac{c_1^3}{3!}x^3 + \frac{c_1^4}{4!}x^4 + \frac{c_1^5}{5!}x^5 + \cdots \\ &= 1 + c_1x + \frac{(c_1x)^2}{2} + \frac{(c_1x)^3}{3!} + \frac{(c_1x)^4}{4!} + \frac{(c_1x)^5}{5!} + \cdots \\ &= e^{c_1x}. \end{aligned}$$

So,

$$\text{if} \quad f(x+y) = f(x)f(y) \quad \text{then} \quad f(x) = e^{c_1x}.$$

Example: Find e^0.

$$e^0 = 1 + 0 + \frac{0^2}{2!} + \frac{0^3}{3!} + \cdots = 1 + 0 + 0 + 0 + \cdots = 1.$$

Example: Explain why $e^{-x} = \dfrac{1}{e^x}$.

$$e^xe^{-x} = e^{x+(-x)} = e^{x-x} = e^0 = 1.$$

Divide both sides by e^x.

$$\text{So} \quad e^{-x} = \frac{1}{e^x}.$$

If k is a positive integer then

$$a^k = \underbrace{a\cdot a\cdot \cdots \cdot a}_{k \text{ factors}}.$$

Thus, if x and y are positive integers then

$$a^xa^y = a^{x+y} \qquad \text{and} \qquad a^1 = a.$$

If $f(x)$ is a function such that $f(x)f(y) = f(x+y)$ then $f(x) = e^{a_1 x}$ for some numer a_1 and so

$$a^x = e^{c_1 x}, \quad \text{for some number } c_1.$$

Since $a^1 = a$,

$$e^{c_1} = e^{c_1 \cdot 1} = a^1 = a, \qquad \text{and so} \qquad c_1 = \ln a.$$

So

$$a^x = e^{x \ln a}.$$

Example: Verify $(e^x)^y = e^{xy}$.

Using the identity $a^x = e^{x \ln a}$,

$$(e^x)^y = e^{y \ln(e^x)} = e^{yx}.$$

Example: Is there a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

whose derivative is itself,

$$\frac{df}{dx} = f \qquad ???$$

If so,

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

must be equal to

$$\frac{df}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \cdots .$$

Comparing terms in these two expressions gives

$$c_1 = c_0, \quad 2c_2 = c_1, \quad 3c_3 = c_2, \quad 4c_4 = c_3, \quad 5c_5 = c_4, \quad 6c_6 = c_5, \quad \ldots$$

So

$$c_2 = \frac{c_0}{2}, \quad 3c_3 = \frac{c_0}{2}, \quad 4c_4 = \frac{c_0}{3 \cdot 2}, \quad 5c_5 = \frac{c_0}{4 \cdot 3 \cdot 2}, \quad \ldots .$$

So

$$\begin{aligned} f(x) &= c_0 + c_0 x + \frac{c_0}{2} x^2 + \frac{c_0^2}{3!} x^3 + \frac{c_0}{4!} x^4 + \frac{c_0}{5!} x^5 + \cdots \\ &= c_0 (1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots) \\ &= c_0 e^x. \end{aligned}$$

So, if $\frac{df}{dx} = f$ then $f = c_0 e^x$.

Example: Verify $e^{y+x} = e^y e^x$.

By the chain rule

$$\frac{d}{dx} e^{2+x} = e^{2+x} \cdot \frac{d(2+x)}{dx} = e^{2+x} \cdot 1 = e^{2+x}.$$

So

$$e^{2+x} = c_0 e^x, \quad \text{for some number } c_0.$$

Since $e^{2+0} = e^2$,

$$c_0 = c_0 e^0 = e^{2+0} = e^2, \qquad \text{and so} \qquad e^{2+x} = e^2 e^x.$$

Similarly,

$$e^{10+x} = e^{10} e^x \;\; \text{and} \;\; e^{642+x} = e^{642} e^x$$

and

$$e^{y+x} = e^y e^x.$$

4 The basic trig identities

Define

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,
\end{aligned}$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Example: Explain why $e^{ix} = \cos x + i \sin x$, if $i^2 = -1$.

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i \cdot i^2x^3}{3!} + \frac{((i^2)^2x^4}{4!} + \frac{i \cdot (i^2)^2x^5}{5!} + \frac{(i^2)^3x^6}{6!} + \frac{i \cdot (i^2)^3x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{i \cdot (-1)x^3}{3!} + \frac{(-1)^2x^4}{4!} + \frac{i \cdot (-1)^2x^5}{5!} + \frac{(-1)^3x^6}{6!} + \frac{i \cdot (-1)^3x^7}{7!} + \cdots, \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots, \\
&= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) \\
&= \cos x + i \sin x.
\end{aligned}$$

Example: Explain why $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.

$$\begin{aligned}\cos(-x) &= 1 - \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} - \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} - \cdots, \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots, \\ &= \cos x,\end{aligned}$$

and

$$\begin{aligned}\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} - \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} - \cdots, \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \frac{x^{13}}{13!} + \cdots, \\ &= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots\right), \\ &= -\sin x.\end{aligned}$$

Example: Explain why $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}1 = e^0 &= e^{ix+(-ix)} \\ &= e^{ix}e^{-ix} = e^{ix}e^{i(-x)} \\ &= (\cos x + i\sin x)(\cos(-x) + i\sin(-x)) \\ &= (\cos x + i\sin x)(\cos x + i(-\sin x)) \\ &= \cos^2 x - i\sin x\cos x + i\sin x\cos x - i^2\sin^2 x \\ &= \cos^2 x - (-1)\sin^2 x \\ &= \cos^2 x + \sin^2 x.\end{aligned}$$

Example: Explain why $\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y, \quad \text{and} \\ \sin(x+y) &= \sin x\cos y + \cos x\sin y.\end{aligned}$

$$\begin{aligned}\cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\ &= e^{ix+iy} = e^{ix}e^{iy} \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y \\ &= \big(\cos x\cos y + (-1)\sin x\sin y\big) + i\big(\cos x\sin y + \sin x\cos y\big).\end{aligned}$$

Comparing terms on each side gives

$$\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y, \qquad \text{and} \\ \sin(x+y) &= \sin x\cos y + \cos x\sin y.\end{aligned}$$

Define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$
$$\sinh x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots,$$
$$\cosh x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \mathrm{sech} x = \frac{1}{\cosh x}, \quad \mathrm{csch} x = \frac{1}{\sinh x}.$$

Example: Explain why $e^x = \cosh x + \sinh x$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \\ &= \cosh x + \sinh x. \end{aligned}$$

Example: Explain why $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

$$\begin{aligned} \cosh(-x) &= 1 + \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} + \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} + \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} + \cdots \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots \\ &= \cosh x, \end{aligned}$$

and

$$\begin{aligned} \sinh(-x) &= (-x) + \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} + \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} + \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} + \cdots \\ &= -x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \frac{x^{11}}{11!} - \frac{x^{13}}{13!} - \cdots \\ &= -\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots\right) \\ &= -\sinh x. \end{aligned}$$

Example: Explain why $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

$$
\begin{aligned}
\tfrac{1}{2}(e^x + e^{-x}) &= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\quad + 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots\big) \\
&= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\quad + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\big) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\
&= \cosh x.
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{2}(e^x - e^{-x}) &= \tfrac{1}{2}\big((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots) \\
&\quad - (1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots)\big) \\
&= \tfrac{1}{2}\big((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots) \\
&\quad - (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots)\big) \\
&= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\quad - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \cdots\big) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \\
&= \sinh x.
\end{aligned}
$$

Example: Explain why $\cosh^2 x - \sinh^2 x = 1$.

$$
\begin{aligned}
1 = e^0 &= e^{x+(-x)} \\
&= e^x e^{-x} \\
&= (\cosh x + \sinh x)(\cosh(-x) + \sinh(-x)) \\
&= (\cosh x + \sinh x)(\cosh x - \sinh x)) \\
&= \cosh^2 x - \sinh x \cosh x + \sinh x \cosh x - \sinh^2 x \\
&= \cosh^2 x - \sinh^2 x.
\end{aligned}
$$

Example: Explain why

$$
\begin{aligned}
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \qquad \text{and} \\
\sinh(x+y) &= 2 \sinh x \cosh y.
\end{aligned}
$$

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y + e^{-x}e^y + e^x e^{-y} + e^{-x}e^{-y}}{4} \\
&\quad + \frac{e^x e^y - e^{-x}e^y - e^x e^{-y} + e^{-x}e^{-y}}{4} \\
&= \frac{2e^x e^y + 2e^{-x}e^{-y}}{4} \\
&= \frac{e^{(x+y)} + e^{-(x+y)}}{2} \\
&= \cosh(x+y),
\end{aligned}$$

and

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y - e^{-x}e^y + e^x e^{-y} - e^{-x}e^{-y}}{4} \\
&\quad + \frac{e^x e^y + e^{-x}e^y - e^x e^{-y} - e^{-x}e^{-y}}{4} \\
&= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} \\
&= \sinh(x+y).
\end{aligned}$$

5 Angles

PICTURE

π is the distance half way around a circle of radius 1. Measure angles according to the distance traveled on a circle of radius 1.

PICTURE

The angle θ is measured by traveling a distance θ on a circle of radius 1. Stretch both x and y to get a circle of radius r.

PICTURE

The distance θ stretches to $r\theta$. Hence, the

$$(\text{arc length along an angle } \theta \text{ on a circle of radius } r) = r\theta.$$

The distance 2π around a circle of radius 1 stretches to $2\pi r$ around a circle of radius r. So the circumference of a circle is $2\pi r$ is the circle is radius r.

To find the area of a circle first approximate with a polygon inscribed in the circle.

PICTURE

the eight triangles form an octagon P_8 in the circle. The area of the octagon is almost the same as the area of the circle. Unwrap the octagon.

$$PICTURE$$

The area of the octagon is the area of the 8 triangles. The area of each triangle is $\frac{1}{2}bh$. So the area of the octagon is $\frac{1}{2}Bh$.

Take the limit as the number of triangles in the interior polygon gets larger and larger (the polygon gets closer and closer to being the circle). Then

$$\begin{aligned}
\text{Area of the circle} &= \lim_{n\to\infty}\Big(\text{area of an } n\text{-sided polygon } P_n\Big) \\
&= \lim_{n\to\infty}\left(\frac{1}{2}Bh\right) \\
&PICTURE \text{total base} \quad \text{height of triangle} \\
&= \frac{1}{2}(2\pi r)(r) \\
&PICTURE \text{length of an unwrapped circle} \quad \text{radius of the circle} \\
&= \pi r^2.
\end{aligned}$$

So the area of a circle is πr^2 if the circle is radius r, and the

$$\begin{aligned}
(\text{area of an arc of angle } \theta \text{ for a circle of radius } r) &= \frac{\theta}{2\pi}\cdot(\text{area of the whole circle}) \\
&= \frac{\theta}{2\pi}\cdot \pi r^2 = \frac{\theta r^2}{2}.
\end{aligned}$$

5.1 Triangles

Example: Verify that

$$x^2 + y^2 = r^2 \qquad \text{for the triangle} \quad PICTURE.$$

The picture

$$PICTURE$$

gives that

$$\begin{aligned}
(x+y)^2 &= (\text{Area of the big square}) \\
&= (\text{Area of the little square}) + (\text{Area of the triangles}) \\
&= r^2 + 4\frac{1}{2}xy = r^2 + 2xy,
\end{aligned}$$

So $x^2 + y^2 = r^2$.

Define

$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \cos\theta = xr.$$

Example: Verify $\sin^2\theta + \cos^2\theta = 1$.

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Measure angles according to distance along a circle of radius 1, starting from $(1,0)$. Then

$$\begin{aligned}
\sin\theta &= y\text{-coordinate of the point at angle } \theta \text{ on a circle of radius 1}, \\
\cos\theta &= x\text{-coordinate of the point at angle } \theta \text{ on a circle of radius 1}.
\end{aligned}$$

This way of looking at $\sin\theta$ and $\cos\theta$ has the advantage that it makes sense for all real numbers θ.

Example: Verify $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

This is pretty obvious from the picture

$$PICTURE$$

Example: Verify $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Since $1^2 + 1^2 = (\sqrt{2})^2$ the hypotenuse of the triangle

$$PICTURE$$

is $\sqrt{2}$. It has angle $\pi/4$ since

$$PICTURE.$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Example: Verify that

$$\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Half of the equilateral triangle

$$PICTURE \qquad \text{is} \qquad PICTURE$$

which has angles $\pi/3$ and $\pi/6$ and hypotenuse $\frac{\sqrt{3}}{2}$ because

$$PICTURE \qquad \text{and} \qquad \left(\frac{1}{2}\right)^2 + \left(\sqrt{3}2\right)^2 = 1.$$

Example: Verify $\cos(x+y) = \cos x\cos y - \sin x\sin y$.

The picture

$$PICTURE$$

has

$$\frac{f}{d} = \sin x, \qquad \frac{\cos(x+y)}{d} = \cos x, \qquad \frac{e}{g} = \sin x, \qquad \frac{\sin y}{g} = \cos x,$$

so that

$$d = \frac{\cos(x+y)}{\cos x}, \qquad g = \frac{\sin y}{\cos x}, \qquad e = g\sin x = \frac{\sin y \sin x}{\cos x}, \qquad f = d\sin x = \frac{cos(x+y)\sin x}{\cos x}.$$

Then

$$d + e = \cos y = \frac{\cos(x+y)}{\cos x} + \frac{\sin y}{\sin x}\cos x = \frac{\cos(x+y) + \sin y\sin x}{\cos x},$$

and so

$$\cos(x+y) = \cos y\cos x - \sin y\sin x.$$

Then

$$\begin{aligned}
\sin(x+y) = f+g &= \frac{\cos(x+y)\sin x}{\cos x} + \frac{\sin y}{\cos x} \\
&= \frac{(\cos x\cos y - \sin x \sin y)\sin x + \sin y}{\cos x} \\
&= \frac{\sin x\cos x\cos y - \sin^2 x\sin y + \sin y}{\cos x} = \frac{\sin x\cos x\cos y + \sin y(1-\sin^2 x)}{\cos x} \\
&= \frac{\sin x\cos x\cos y + \sin y\cos^2 x}{\cos x} = \sin x\cos y + \sin y\cos x.
\end{aligned}$$

5.2 Trig functions

$\sin\theta$ is the y-coordinate of a point at distance θ on a circle of radius 1,

$\cos\theta$ is the x-coordinate of a point at distance θ on a circle of radius 1,

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$,

$\sec\theta = \dfrac{1}{\cos\theta}$,

$\csc\theta = \dfrac{1}{\sin\theta}$,

Since the equation of a circle of radius 1 is $x^2+y^2=1$ this forces

$$\sin^2\theta + \cos^2\theta = 1.$$

The pictures

$$PICTURE \qquad \text{and} \qquad PICTURE$$

show that

$$\sin(-\theta) = -\sin\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\theta.$$

Also

$$PICTURE \qquad \text{and} \qquad PICTURE$$

show that

$$\begin{matrix} \sin 0 &= 0 \\ \cos 0 &= 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} \sin\frac{\pi}{2} &= 1, \\ \cos\frac{\pi}{2} &= 0. \end{matrix}$$

Draw the graphs

$$PICTURE \qquad \text{and} \qquad PICTURE,$$

by seeing how the x and y coordinates change as you walk around the circle.

There are five trig identities to remember:

$$\begin{gathered}
\sin(x+y) = \sin x\cos y + \cos x\sin y, \\
\cos(x+y) = \cos x\cos y - \sin x\sin y, \\
\sin^2 x + \cos^2 x = 1, \\
\sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x,
\end{gathered}$$

As well as the two triangles

$$PICTURE \qquad \text{and} \qquad PICTURE.$$

From these triangles,

$$\begin{array}{ll} \sin\frac{\pi}{6} = \frac{1}{2} & \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} & \cos\frac{\pi}{3} = \frac{1}{2} \\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{array}$$

Since the only trig identities I remember are identities for sines and cosines I usually verify trig identities by first writing them completely in terms of sines and cosines.

Example. Verify $\dfrac{\sec B}{\cos B} - \dfrac{\tan B}{\cot B} = 1$.

$$\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} = \frac{\left(\frac{1}{\cos B}\right)}{\cos B} - \frac{\left(\frac{\sin B}{\cos B}\right)}{\left(\frac{\cos B}{\sin B}\right)}$$

$$= \frac{1}{\cos^2 B} - \frac{\sin^2 B}{\cos^2 B} = \frac{1-\sin^2 B}{\cos^2 B} = \frac{\cos^2 B}{\cos^2 B} = 1.$$

Example. Verify $\cot\alpha - \cot\beta = \dfrac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$.

$$\text{Left Hand Side} = \cot\alpha - \cot\beta = \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta}$$

$$= \frac{\cos\alpha\sin\beta - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$

$$\text{Right Hand Side} = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos(-\alpha)+\cos\beta\sin(-\alpha)}{\sin\alpha\sin\beta}$$

$$= \frac{\sin\beta\cos\alpha + \cos\beta(-\sin\alpha)}{\sin\alpha\sin\beta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta}.$$

So

$$\text{Left Hand Side} = \text{Right Hand Side}.$$

Example. Verify $\dfrac{\tan A - \sin A}{\sec A} = \dfrac{\sin^3 A}{1+\cos A}$.

$$\frac{\tan A - \sin A}{\sec A} \stackrel{?}{=} \frac{\sin^3 A}{1+\cos A}$$

So $\quad (1+\cos A)(\tan A - \sin A) \stackrel{?}{=} \sin^3 A\sec A.$

So $\quad \tan A + \cos A\tan A - \sin A - \sin A\cos A \stackrel{?}{=} \sin^3 A\sec A.$

So $\quad \dfrac{\sin A}{\cos A} + \cos A\left(\dfrac{\sin A}{\cos A}\right) - \sin A - \sin A\cos A \stackrel{?}{=} \sin^3 A\left(\dfrac{1}{\cos A}\right).$

So $\quad \dfrac{\sin A}{\cos A} + \sin A - \sin A - \sin A\cos A \stackrel{?}{=} \sin^3 A\left(\dfrac{1}{\cos A}\right).$

So $\quad \dfrac{\sin A - \sin A\cos^2 A}{\cos A} \stackrel{?}{=} \dfrac{\sin^3 A}{\cos A}$

So $\quad \sin A - \sin A\cos^2 A \stackrel{?}{=} \dfrac{\sin^3 A}{\cos A}.$

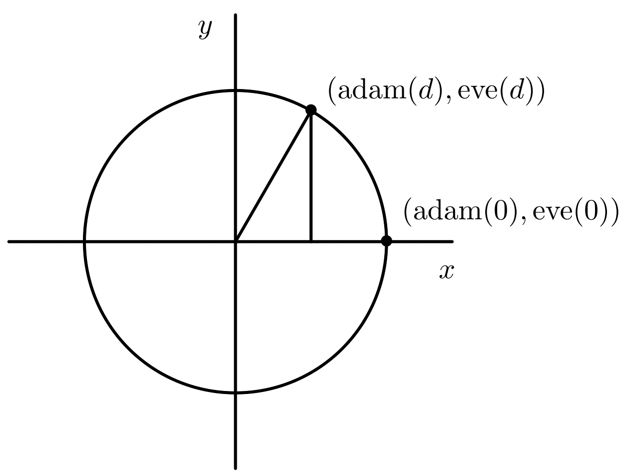
So $\quad 1 - \cos^2 A \stackrel{?}{=} \sin^2 A.$

YES, because $\sin^2 A + \cos^2 A = 1$.

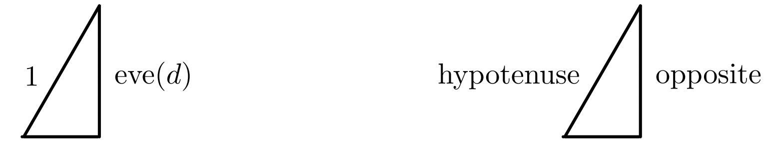
and so

adam(t) = x-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0), and

eve(t) = y-coordinate of the point on a circle of radius 1 which is distance d from the point (1,0).



The triangle in this picture is



and so

$$\text{adam}(d) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \text{eve}(d) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

for a right triangle with angle d.

Example. Verify $\sin 3x = 3\sin x\cos^2 x - \sin^3 x$ and $\cos 3x = \cos^3 x - 3\sin^2 x\cos x$

$$\begin{aligned}
e^{i3x} &= \cos 3x + i\sin 3x \\
&= (e^{ix})^3 = (\cos x + i\sin x)^3 \\
&= (\cos^2 x + 2i\sin x\cos x + i^2\sin^2 x)(\cos x + i\sin x) \\
&= \cos^3 x + i\sin x\cos^2 x + 2i\sin x\cos^2 x - 2\sin^2 x\cos x \\
&\qquad - \sin^2 x\cos x - i\sin^3 x \\
&= \cos^3 x + 3i\sin x\cos^2 x - 3\sin^2 x\cos x - i\sin^3 x \\
&= \cos^3 x - 3\sin^2 x\cos x + i(3\sin x\cos^2 x - \sin^3 x).
\end{aligned}$$

So

$$\begin{aligned}
\cos 3x + i\sin 3x &= \cos^3 x - 3\sin^2 x\cos x \\
&\qquad + i(3\sin x\cos^2 x - \sin^3 x).
\end{aligned}$$

So

$$\cos 3x = \cos^3 x - 3\sin^2 x\cos x$$

and

$$\sin 3x = 3\sin x\cos^2 x - \sin^3 x.$$

6 Inverse functions

$\sqrt{x}$ is the function that undoes x^2. This means that

$$\sqrt{x^2} = x \qquad \text{and} \qquad (\sqrt{x})^2 = x.$$

$\ln x$ is the function that undoes e^x. This means that

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x.$$

$\sin^{-1} x$ is the function that undoes $\sin x$. This means that

$$\sin^{-1}(\sin x) = x \qquad \text{and} \qquad \sin(\sin^{-1} x) = x.$$

$\cos^{-1} x$ is the function that undoes $\cos x$. This means that

$$\cos^{-1}(\cos x) = x \qquad \text{and} \qquad \cos(\cos^{-1} x) = x.$$

$\tan^{-1} x$ is the function that undoes $\tan x$. This means that

$$\tan^{-1}(\tan x) = x \qquad \text{and} \qquad \tan(\tan^{-1} x) = x.$$

$\cot^{-1} x$ is the function that undoes $\cot x$. This means that

$$\cot^{-1}(\cot x) = x \qquad \text{and} \qquad \cot(\cot^{-1} x) = x.$$

$\sec^{-1} x$ is the function that undoes $\sec x$. This means that

$$\sec^{-1}(\sec x) = x \qquad \text{and} \qquad \sec(\sec^{-1} x) = x.$$

$\csc^{-1} x$ is the function that undoes $\csc x$. This means that

$$\csc^{-1}(\csc x) = x \qquad \text{and} \qquad \csc(\csc^{-1} x) = x.$$

$\log_a x$ is the function that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin 32}) = \sqrt{7}\pi i \sin 32 \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32.$$

WARNING: $\sin^{-1} x$ is VERY DIFFERENT from $(\sin x)^{-1}$. For example,

$$\sin^{-1} 0 = \sin^{-1}(\sin 0) = 0, \qquad \text{BUT} \qquad (\sin 0)^{-1} = \frac{1}{\sin 0} = \frac{1}{0} = \text{UNDEFINED}.$$

Example: Explain why $\ln 1 = 0$.

$$\ln 1 = \ln(e^0) = 0.$$

Example: Explain why $\ln(ab) = \ln a + \ln b$.

$$\ln(ab) = \ln(e^{\ln a} \cdot e^{\ln b}) = \ln(e^{\ln a + \ln b}) = \ln a + \ln b.$$

Example: Explain why $\ln\left(\frac{1}{a}\right) = -\ln a$.

$$\ln\left(\frac{1}{a}\right) = \ln\left(\frac{1}{e^{\ln a}}\right) = \ln\left(e^{-\ln a}\right) = -\ln a.$$

Example: Explain why $\ln\left(a^b\right) = b \ln a$.

$$\ln(a^b) = \ln\left(\left(e^{\ln a}\right)^b\right) = \ln\left(e^{b \ln a}\right) = b \ln a.$$

Thus

$$\begin{array}{lll} e^0 = 1 & \text{turns into} & \ln 1 = 0, \\ e^x e^y = e^{x+y} & \text{turns into} & \ln(ab) = \ln a + \ln b, \\ e^{-x} = \dfrac{1}{e^x} & \text{turns into} & \ln\left(\dfrac{1}{a}\right) = -\ln a, \quad \text{and} \\ (e^x)^y = e^{yx} & \text{turns into} & \ln(a^b) = b \ln a. \end{array}$$

7 Derivatives

A **function** eats a number, chews on it, and spits out another number.

PICTURE

A **constant function** always spits out the same number, no matter what the input is.

Example: $f(x) = 2$.

PICTURE

We call this function 2. So, 2 *sometimes means the number* 2, and *sometimes means the function* 2.

A **derivative** eats a function, chews on it, and spits out another function.

PICTURE

The derivative $\frac{d}{dx}$ knows what to spit out by always following the rules:

(1) $\frac{dx}{dx} = 1$,

(2) $\frac{d(cf)}{dx} = c\frac{df}{dx}$, if c does not change when x changes,

(3) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$,

(4) $\frac{d(fg)}{dx} = f\,\frac{dg}{dx} + \frac{df}{dx}\,g$.

Example: Find $\frac{dy}{dx}$ if $y = 5x$. $\frac{dy}{dx} = \frac{d(5x)}{dx} = 5\,\frac{dx}{dx} = 5 \cdot 1 = 5$.

Example: Find $\frac{dy}{dx}$ if $y = \pi x$. $\frac{dy}{dx} = \frac{d(\pi x)}{dx} = \pi\,\frac{dx}{dx} = \pi \cdot 1 = \pi$.

Example: Find $\frac{dy}{dx}$ if $y = 1$.

$$\frac{dy}{dx} = \frac{d1}{dx} = \frac{d(1 \cdot 1)}{dx} = 1 \cdot \frac{d1}{dx} + \frac{d1}{dx} \cdot 1 = \frac{d1}{dx} + \frac{d1}{dx}.$$

Subtract $\frac{d1}{dx}$ from both sides. So $\frac{d1}{dx} = 0$.

Example: Find $\frac{dy}{dx}$ if $y = 5$.

$$\frac{dy}{dx} = \frac{d5}{dx} = \frac{d(5 \cdot 1)}{dx} = 5 \cdot \frac{d1}{dx} = 5 \cdot 0 = 0.$$

Example: Find $\frac{dy}{dx}$ if $y = 6342$.

$$\frac{dy}{dx} = \frac{d\,6342}{dx} = \frac{d(6342 \cdot 1)}{dx} = 6342 \cdot \frac{d1}{dx} = 6342 \cdot 0 = 0.$$

Example: Find $\frac{dc}{dx}$ if c is a constant.

$$\frac{dc}{dx} = \frac{d(c \cdot 1)}{dx} = c \cdot \frac{d1}{dx} = c \cdot 0 = 0.$$

Example: Find $\frac{dy}{dx}$ if $y = 3x + 12$.

$$\frac{dy}{dx} = \frac{d(3x+12)}{dx} = \frac{d(3x)}{dx} + \frac{d(12)}{dx} = 3\frac{dx}{dx} + 0 = 3 \cdot 1 + 0 = 3.$$

Example: Find $\frac{dy}{dx}$ if $y = x^2$.

$$\frac{dy}{dx} = \frac{dx^2}{dx} = \frac{d(x \cdot x)}{dx} = x\,\frac{dx}{dx} + \frac{dx}{dx}\,x = x \cdot 1 + 1 \cdot x = 2x.$$

Example: Find $\frac{dy}{dx}$ if $y = x^3$.

$$\frac{dy}{dx} = \frac{dx^3}{dx} = \frac{d(x^2 \cdot x)}{dx} = x^2\,\frac{dx}{dx} + \frac{dx^2}{dx}\,x = x^2 \cdot 1 + 2x \cdot x = 3x^2.$$

Example: Find $\frac{dy}{dx}$ if $y = x^4$.

$$\frac{dy}{dx} = \frac{dx^4}{dx} = \frac{d(x^3 \cdot x)}{dx} = x^3\,\frac{dx}{dx} + \frac{dx^3}{dx}\,x = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3.$$

... and we keep on going ...

Example: Find $\frac{dy}{dx}$ if $y = x^{6342}$.

$$\frac{dy}{dx} = \frac{dx^{6342}}{dx} = \frac{d(x^{6341} \cdot x)}{dx} = x^{6341}\,\frac{dx}{dx} + \frac{dx^{6341}}{dx}\,x = x^{6341} \cdot 1 + 6341x^{6340} \cdot x = 6342x^{6341}.$$

… and we keep on going …

Example: Find $\frac{dx^n}{dx}$ for $n = 1, 2, 3, \ldots$.

$$\begin{aligned}\frac{dy}{dx} = \frac{dx^n}{dx} = \frac{d(x^{n-1}\cdot x)}{dx} &= x^{n-1}\,\frac{dx}{dx} + \frac{dx^{n-1}}{dx}\,x \\ &= x^{n-1}\cdot 1 + (n-1)x^{n-2}\cdot x, \qquad \text{since we already found } \frac{dx^{n-1}}{dx} = (n-1)x^{n-2}, \\ &= nx^{n-1}.\end{aligned}$$

and thus we have found $\quad \frac{dx^n}{dx} = nx^{n-1}, \quad$ for all positive integers n. (Amazing!)

Example: Find $\frac{dx^n}{dx}$ for $n = 0$.

$$\frac{dy}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0 = 0x^{-1} = 0x^{0-1}.$$

Example: Find $\frac{dx^{-6342}}{dx}$.

$$\frac{dx^{-6342}\cdot x^{6342}}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-6342}\cdot x^{6342}}{dx} = x^{-6342}\frac{dx^{6342}}{dx} + \frac{dx^{-6342}}{dx}\cdot x^{6342} = x^{-6342}\cdot 6342x^{6341} + \frac{dx^{-6342}}{dx}\cdot x^{6342}.$$

So

$$0 = x^{-6342}\cdot 6342x^{6341} + \frac{dx^{-6342}}{dx}\cdot x^{6342}.$$

Solve for $\frac{dx^{-6342}}{dx}$.

$$\frac{dx^{-6342}}{dx} = -6342x^{-1}x^{-6342} = (-6342)x^{-6343}.$$

Example: Find $\frac{dx^{-n}}{dx}$ for $n = 1, 2, 3, \ldots$.

$$\frac{dx^{-n}\cdot x^n}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-n}\cdot x^n}{dx} = x^{-n}\frac{dx^n}{dx} + \frac{dx^{-n}}{dx}\cdot x^n = x^{-n}\cdot nx^{n-1} + \frac{dx^{-n}}{dx}\cdot x^n.$$

So

$$0 = x^{-n} \cdot nx^{n-1} + \frac{dx^{-n}}{dx} \cdot x^n.$$

Solve for $\frac{dx^{-n}}{dx}$.

$$\frac{dx^{-n}}{dx} = -nx^{-1}x^{-n} = (-n)x^{-n-1}.$$

and thus we have found $\quad \frac{dx^n}{dx} = nx^{n-1}, \quad$ for *all* integers n. (AMAZING!)

Example: Let $y = 3x^3 + 5x^2 + 2x + 7$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(3x^3 + 5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(2x)}{dx} + \frac{d7}{dx} \\
&= 3\,\frac{dx^3}{dx} + 5\,\frac{dx^2}{dx} + 2\,\frac{dx}{dx} + 7\,\frac{d1}{dx} \\
&= 3 \cdot 3x^2 + 5 \cdot 2x + 2 \cdot 1 + 7 \cdot 0 = 9x^2 + 10x + 2.
\end{aligned}$$

Example: Let $y = -7x^{-13} + 5x^{-7} + (6 + 2i)x^{38}$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(-7x^{-13} + 5x^{-7} + (6 + 2i)x^{38})}{dx} \\
&= \frac{d(-7x^{-13})}{dx} + \frac{d(5x^{-7})}{dx} + \frac{d((6 + 2i)x^{38})}{dx} \\
&= -7\,\frac{dx^{-13}}{dx} + 5\,\frac{dx^{-7}}{dx} + (6 + 2i)\,\frac{dx^{38}}{dx} \\
&= (-7) \cdot (-13)x^{-13-1} + 5(-7)x^{-7-1} + (6 + 2i) \cdot 38 \cdot x^{38-1} \\
&= 91x^{-14} - 35x^{-8} + (228 + 76i)x^{37}.
\end{aligned}$$

8 The chain rule

There are **different kinds of derivatives**:

Derivative with respect to x

$$f \longrightarrow \tfrac{d}{dx} \longrightarrow \tfrac{df}{dx}$$

This one satisfies

$$\frac{dx}{dx} = 1,$$

$$\frac{d(cf)}{dx} = c\frac{df}{dx}, \quad \text{if } c \text{ is a constant,}$$

$$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx},$$

$$\frac{d(yz)}{dx} = y\,\frac{dz}{dx} + \frac{dy}{dx}\,z.$$

Derivative with respect to g

$$f \longrightarrow \tfrac{d}{dg} \longrightarrow \tfrac{df}{dg}$$

This one satisfies

$$\frac{dg}{dg} = 1,$$

$$\frac{d(cf)}{dg} = c\frac{df}{dg}, \quad \text{if } c \text{ is a constant,}$$

$$\frac{d(y+z)}{dg} = \frac{dy}{dg} + \frac{dz}{dg},$$

$$\frac{d(yz)}{dg} = y\,\frac{dz}{dg} + \frac{dy}{dg}\,z.$$

What is the relation between $\dfrac{df}{dx}$ **and** $\dfrac{df}{dg}$ **?**

$$\longrightarrow \frac{d}{dg} \longrightarrow \qquad\qquad \longrightarrow \frac{d}{dx} \longrightarrow$$

$$\frac{dg^0}{dg} = \frac{d1}{dg} = 0, \qquad\qquad \frac{dg^0}{dx} = \frac{d1}{dx} = 0,$$

$$\frac{dg}{dg} = 1, \qquad\qquad \frac{dg}{dx} = \frac{dg}{dx},$$

$$\begin{aligned}\frac{dg^2}{dg} &= \frac{dg \cdot g}{dg}\\ &= g\frac{dg}{dg} + \frac{dg}{dg}g\\ &= g + g = 2g,\end{aligned} \qquad\qquad \begin{aligned}\frac{dg^2}{dx} &= \frac{dg \cdot g}{dx}\\ &= g\frac{dg}{dx} + \frac{dg}{dx}g\\ &= 2g\frac{dg}{dx},\end{aligned}$$

$$\begin{aligned}\frac{dg^3}{dg} &= \frac{dg^2 \cdot g}{dg}\\ &= g^2\frac{dg}{dg} + \frac{dg^2}{dg}g\\ &= g^2 + 2g \cdot g = 3g^2,\end{aligned} \qquad\qquad \begin{aligned}\frac{dg^3}{dx} &= \frac{dg^2 \cdot g}{dx}\\ &= g^2\frac{dg}{dx} + \frac{dg^2}{dx}g\\ &= g^2\frac{dg}{dx} + 2g\frac{dg}{dx}g\\ &= g^2\frac{dg}{dx} + 2g^2\frac{dg}{dx}\\ &= 3g^2\frac{dg}{dx},\end{aligned}$$

$$\begin{aligned}\frac{dg^4}{dg} &= \frac{dg^3 \cdot g}{dg}\\ &= g^3\frac{dg}{dg} + \frac{dg^3}{dg}g\\ &= g^3 + 3g^2 \cdot g = 4g^3,\end{aligned} \qquad\qquad \begin{aligned}\frac{dg^4}{dx} &= \frac{dg^3 \cdot g}{dx}\\ &= g^3\frac{dg}{dx} + \frac{dg^3}{dx}g\\ &= g^3\frac{dg}{dx} + 3g^2\frac{dg}{dx}g\\ &= g^3\frac{dg}{dx} + 3g^3\frac{dg}{dx}\\ &= 4g^3\frac{dg}{dx},\end{aligned}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{dg^{6342}}{dg} = 6342g^{6341}, \qquad\qquad \frac{dg^{6342}}{dx} = 6342g^{6341}\frac{dg}{dx},$$

$$\begin{aligned}\frac{d(3g^2+2g+7)}{dg} &= \frac{d(3g^2)}{dg} + \frac{d(2g)}{dg} + \frac{d7}{dg} \\ &= 3\frac{dg^2}{dg} + 2\frac{dg}{dg} + 0 \\ &= 3\cdot 2g + 2\cdot 1 \\ &= 6g+2,\end{aligned}$$

$$\begin{aligned}\frac{d(3g^2+2g+7)}{dx} &= \frac{d(3g^2)}{dx} + \frac{d(2g)}{dx} + \frac{d7}{dx} \\ &= 3\frac{dg^2}{dx} + 2\frac{dg}{dx} + 0 \\ &= 3\cdot 2g\frac{dg}{dx} + 2\frac{dg}{dx} \\ &= (6g+2)\frac{dg}{dx},\end{aligned}$$

Thus, we are seeing that

$$\frac{df}{dx} = \frac{df}{dg}\,\frac{dg}{dx}\,, \qquad \text{which is the chain rule.}$$

Example: Find $\dfrac{dy}{dx}$ when $y = (2x-5)^2$.

If $g = 2x-5$ then $y = g^2$.

$$\frac{dy}{dx} = \frac{dy}{dg}\,\frac{dg}{dx} = \frac{dg^2}{dg}\,\frac{d(2x-5)g}{dx} = 2g(2-0) = 2(2x-5)\cdot 2$$

$$= 4(2x-5) = 8x-20.$$

Example: Find $\dfrac{dy}{dx}$ when $y = (3x-4)^3$.

If $g = 3x-4$ then $y = g^3$.

$$\frac{dy}{dx} = \frac{dy}{dg}\,\frac{dg}{dx} = \frac{dg^3}{dg}\,\frac{d(3x-4)g}{dx} = 3g^2(3-0) = 9(3x-4)^2$$

$$= 9(9x^2-24x+16) = 81x^2-72x+144.$$

Example: Find $\dfrac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$.

$$\frac{dy}{dx} = \frac{d(2x-5)^2(3x-4)^3}{dx} = (2x-5)^2\,\frac{d(3x-4)^3}{dx} + \frac{d(2x-5)^2}{dx}\,(3x-4)^3$$

$$= (2x-5)^2\cdot 3(3x-4)^2\cdot 3 + 2(2x-5)\cdot 2(3x-4)^3$$

$$= (2x-5)(3x-4)^2(9(2x-5)+4(3x-4)) = (2x-5)(3x-4)^2(30x-61).$$

Example: Find $\dfrac{d\,x^{m/n}}{dx}$ when m and n are integers, $n \neq 0$.

$$\frac{d(x^{m/n})^n}{dx} = \frac{dx^m}{dx} = mx^{m-1}. \qquad \text{On the other hand} \qquad \frac{d(x^{m/n})^n}{dx} = n\big(x^{m/n}\big)^{n-1}\frac{dx^{m/n}}{dx}.$$

So $\quad mx^{m-1} = n\big(x^{m/n}\big)^{n-1}\dfrac{dx^{m/n}}{dx} \quad$ and we can solve for $\quad \dfrac{dx^{m/n}}{dx}$.

$$\frac{dx^{m/n}}{dx} = \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n-1}} = \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n}\big(x^{m/n}\big)^{-1}}$$

$$= \frac{mx^{m-1}}{nx^m\ \frac{1}{x^{m/n}}} = \left(\frac{m}{n}\right)x^{-1}x^{m/n} = \left(\frac{m}{n}\right)x^{(m/n)-1}.$$

Example: Find $\dfrac{dy}{dx}$ when $y = \dfrac{x}{\sqrt{1-2x}}$.

$$\frac{dy}{dx} = \frac{d\ \dfrac{x}{\sqrt{1-2x}}}{dx} = \frac{d\ x\big(\sqrt{1-2x}\big)^{-1}}{dx} = \frac{d\ x\big((1-2x)^{1/2}\big)^{-1}}{dx}$$

$$= \frac{d\ x(1-2x)^{-(1/2)}}{dx} = x\,\frac{d\ (1-2x)^{-(1/2)}}{dx} + \frac{dx}{dx}\,(1-2x)^{-(1/2)}$$

$$= x\big(-\tfrac{1}{2}\big)(1-2x)^{-3/2}\,\frac{d\ (1-2x)}{dx} + 1\cdot\frac{1}{\sqrt{1-2x}}$$

$$= \frac{-x}{2(1-2x)^{3/2}}\cdot(-2) + \frac{1}{(1-2x)^{1/2}} = \frac{x+1-2x}{(1-2x)^{3/2}} = \frac{1-x}{(1-2x)^{3/2}}.$$

Example: Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$.

$$\frac{dy}{dx} = \frac{d\,\dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}}{dx} = \frac{d\,\dfrac{(1+x^2)^{1/2}}{(1-x^2)^{1/2}}}{dx} = \frac{d\left(\dfrac{1+x^2}{1-x^2}\right)^{1/2}}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{(1/2)-1} \frac{d\left(\dfrac{1+x^2}{1-x^2}\right)}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{-(1/2)} \frac{d\,(1+x^2)(1-x^2)^{-1}}{dx}$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \left((1+x^2)\frac{d\,(1-x^2)^{-1}}{dx} + \frac{d\,(1+x^2)}{dx}(1-x^2)^{-1}\right)$$

$$= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \left((1+x^2)(-1)(1-x^2)^{-2}\frac{d\,(1-x^2)^{-1}}{dx} + 2x(1-x^2)^{-1}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \left(\frac{(-1)(1+x^2)(-2x)}{(1-x^2)^2} + \frac{2x}{1-x^2}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \left(\frac{2x(1+x^2)}{(1-x^2)^2} + \frac{2x(1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \left(\frac{2x(1+x^2+1-x^2)}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{(1+x^2)^{1/2}}\cdot\frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{1/2}(1-x^2)^{3/2}}.$$

Example: Differentiate $\dfrac{x^2}{1+x^2}$ with respect to x^2.

This is the same problem as:

Find $\dfrac{dz}{dp}$ when $z = \dfrac{x^2}{1+x^2}$ and $p = x^2$.

Since $\dfrac{dz}{dx} = \dfrac{dz}{dp}\dfrac{dp}{dx}$, $\qquad \dfrac{dz}{dp} = \dfrac{(dz/dx)}{(dp/dx)}$.

So

$$\frac{dz}{dp} = \frac{\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)}{\frac{d}{dx}(x^2)} = \frac{\frac{d\ x^2(1+x^2)^{-1}}{dx}}{\frac{d\ x^2}{dx}} = \frac{x^2\frac{d\ (1+x^2)^{-1}}{dx} + \frac{dx^2}{dx}(1+x^2)^{-1}}{2x}$$

$$= \frac{x^2(-1)(1+x^2)^{-2}\frac{d\ (1+x^2)}{dx} + 2x(1+x^2)^{-1}}{2x}$$

$$= \frac{\frac{-x^2}{(1+x^2)^2}\cdot 2x + \frac{2x}{1+x^2}}{2x} = \frac{-x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$

$$= \frac{-x^2+1+x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2}.$$

Example: Find $\frac{dy}{dx}$ when $x^4 + y^4 = 4a^2x^2y^2$.

$$\frac{d\,(x^4+y^4)}{dx} = \frac{d\,(4a^2x^2y^2)}{dx}. \qquad \text{So} \quad \frac{dx^4}{dx} + \frac{dy^4}{dx} = 4a^2\frac{dx^2y^2}{dx}.$$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2\frac{dy^2}{dx} + \frac{dx^2}{dx}y^2\right).$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right)$
$\quad = 4a^2x^2 2y\frac{dy}{dx} + 4a^2 2xy^2.$

So $\quad 4x^3 - 4a^2 2xy^2 = 4a^2x^2 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}.$

So $\quad 4x^3 - 4a^2 2xy^2 = \left(4a^2x^2 2y - 4y^3\right)\frac{dy}{dx}.$

So $\quad \frac{4x^3 - 4a^2 2xy^2}{4a^2x^2 2y - 4y^3} = \frac{dy}{dx}.$

So $\quad \frac{dy}{dx} = \frac{x^3 - 2a^2xy^2}{2a^2x^2y - y^3}.$

All we did is take the derivative of both sides and then solve for $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$.

Since $y = \frac{3at^2}{1+t^3} = \left(\frac{3at}{1+t^3}\right)t = xt, \qquad \frac{dy}{dx} = x\frac{dt}{dx} + \frac{dx}{dx}\cdot t = x\frac{dt}{dx} + t.$

What is $\frac{dt}{dx}$??

Since $\frac{dx}{dx} = \frac{dx}{dt}\frac{dt}{dx}$, $\qquad \frac{dt}{dx} = \frac{(dx/dx)}{(dx/dt)} = \frac{1}{dx/dt}$.

So

$$\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)} = \frac{1}{\frac{d\ (3at)(1+t^3)^{-1}}{dt}}$$

$$= \frac{1}{3at(-1)(1+t^3)^{-2}\frac{d(1+t^3)}{dt} + 3a(1+t^3)^{-1}}$$

$$= \frac{1}{\frac{-3at}{(1+t^3)^2}3t^2 + \frac{3a}{1+t^3}} = \frac{1}{\frac{-9at^3+3a(1+t^3)}{(1+t^3)^2}}$$

$$= \frac{(1+t^3)^2}{-9at^3+3a(1+t^3)} = \frac{(1+t^3)^2}{3a-6at^3}.$$

So

$$\frac{dy}{dx} = x\frac{dt}{dx} + t = \frac{3at}{1+t^3}\frac{(1+t^3)^2}{3a(1-2t^3)} + t$$

$$= \frac{t(1+t^3)}{1-2t^3} + \frac{t(1-2t^3)}{1-2t^3} = \frac{t+t^4+t-2t^4}{1-2t^3} = \frac{2t-t^4}{1-2t^3}$$

9 Derivatives of trig functions

Define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Example: Find $\frac{de^x}{dx}$.

$$\begin{aligned}
\frac{de^x}{dx} &= \frac{d}{dx}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\big) \\
&= 0 + 1 + \frac{1}{2!}2x + \frac{1}{3!}3x^2 + \frac{1}{4!}4x^3 + \frac{1}{5!}5x^4 + \frac{1}{6!}6x^5 + \frac{1}{7!}7x^6 + \cdots \\
&= 1 + \frac{1}{2}2x + \frac{1}{3\cdot 2!}3x^2 + \frac{1}{4\cdot 3!}4x^3 + \frac{1}{5\cdot 4!}5x^4 + \frac{1}{6\cdot 5!}6x^5 + \frac{1}{7\cdot 6!}7x^6 + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&= e^x.
\end{aligned}$$

Example: Find $\frac{d\sin x}{dx}$.

$$\begin{aligned}
\frac{d\sin x}{dx} &= \frac{d}{dx}\big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots\big) \\
&= 1 - \frac{1}{3!}3x^2 + \frac{1}{5!}5x^4 - \frac{1}{7!}7x^6 + \frac{1}{9!}9x^8 - \frac{1}{11!}11x^{10} + \frac{1}{13!}13x^{12} - \cdots \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \cdots \\
&= \cos x.
\end{aligned}$$

Example: Find $\frac{d\cos x}{dx}$.

$$\begin{aligned}
\frac{d\cos x}{dx} &= \frac{d}{dx}\big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots\big) \\
&= 0 - \frac{1}{2!}2x + \frac{1}{4!}4x^3 - \frac{1}{6!}6x^5 + \frac{1}{8!}8x^7 - \frac{1}{10!}10x^9 + \frac{1}{12!}12x^11 - \cdots \\
&= -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \frac{1}{9!}x^9 + \frac{1}{11!}1x^11 - \cdots \\
&= -\big(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}1x^11 + \cdots\big) \\
&= -\sin x.
\end{aligned}$$

Example: Find $\frac{d\tan x}{dx}$.

$$\begin{aligned}
\frac{d\tan x}{dx} &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\left(\sin x(\cos x)^{-1}\right) \\
&= \sin x\frac{d(\cos x)^{-1}}{dx} + \frac{d\sin x}{dx}(\cos x)^{-1} \\
&= \sin x(-1)(\cos x)^{-2}\frac{d\cos x}{dx} + \cos x \cdot \frac{1}{\cos x} \\
&= -\frac{\sin x}{\cos^2 x}(-\sin x) + 1 = \frac{\sin^2 x}{\cos^2 x} + 1 \\
&= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

Example: Find $\dfrac{d\sec x}{dx}$.

$$\begin{aligned}\frac{d\sec x}{dx} &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left((\cos x)^{-1}\right) = (-1)(\cos x)^{-2}\frac{d\cos x}{dx} \\ &= -\frac{1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x\sec x.\end{aligned}$$

Example: Find $\dfrac{d\csc x}{dx}$.

$$\begin{aligned}\frac{d\csc x}{dx} &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}\left((\sin x)^{-1}\right) = (-1)(\sin x)^{-2}\frac{d\sin x}{dx} \\ &= -\frac{1}{\sin^2 x}(\cos x) = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = -\cot x\csc x.\end{aligned}$$

Example: Find $\dfrac{d\cot x}{dx}$.

$$\begin{aligned}\frac{d\cot x}{dx} &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{d}{dx}\left(\cos x(\sin x)^{-1}\right) \\ &= \cos x\frac{d(\sin x)^{-1}}{dx} + \frac{d\cos x}{dx}(\sin x)^{-1} \\ &= \cos x(-1)(\sin x)^{-2}\frac{d\sin x}{dx} + -(\sin x)\cdot\frac{1}{\sin x} \\ &= -\frac{\cos x}{\sin^2 x}\cdot\cos x - 1 = \frac{-\cos^2 x}{\sin^2 x} - 1 \\ &= \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = \csc^2 x.\end{aligned}$$

Example: Find $\dfrac{d\ \sinh x}{dx}$.

$$\begin{aligned}\frac{d\ \sinh x}{dx} &= \frac{d}{dx}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots\right) \\ &= 1 + \frac{1}{3!}3x^2 + \frac{1}{5!}5x^4 + \frac{1}{7!}7x^6 + \frac{1}{9!}9x^8 + \frac{1}{11!}11x^{10} + \frac{1}{13!}13x^{12} + \cdots \\ &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} + \cdots \\ &= \cosh x.\end{aligned}$$

Example: Find $\dfrac{d\ \cosh x}{dx}$.

$$\begin{aligned}\frac{d\ \cosh x}{dx} &= \frac{d}{dx}\big(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}+\frac{x^{10}}{10!}+\frac{x^{12}}{12!}+\cdots\big)\\ &= 0+\frac{1}{2!}2x+\frac{1}{4!}4x^3+\frac{1}{6!}6x^5+\frac{1}{8!}8x^7+\frac{1}{10!}10x^9+\frac{1}{12!}12x^11+\cdots\\ &= x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\frac{1}{7!}x^7+\frac{1}{9!}x^9+\frac{1}{11!}1x^11+\cdots\\ &= \sinh x.\end{aligned}$$

Example: Find $\frac{d\ \tanh x}{dx}$.

$$\begin{aligned}\frac{d\ \tanh x}{dx} &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)=\frac{d}{dx}\left(\sinh x(\cosh x)^{-1}\right)\\ &= \sinh x\frac{d(\cosh x)^{-1}}{dx}+\frac{d\sinh x}{dx}(\cosh x)^{-1}\\ &= \sinh x(-1)(\cosh x)^{-2}\frac{d\cosh x}{dx}+\cosh x\cdot\frac{1}{\cosh x}\\ &= -\frac{\sinh x}{\cosh^2 x}\cdot\sinh x+1=-\frac{\sinh^2 x}{\cosh^2 x}+1\\ &= \frac{-\sinh^2 x+\cosh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x}=\text{sech}^2 x.\end{aligned}$$

Example: Find $\frac{d\ \text{sech}x}{dx}$.

$$\begin{aligned}\frac{d\ \text{sech}x}{dx} &= \frac{d}{dx}\left(\frac{1}{\cosh x}\right)=\frac{d}{dx}\left((\cosh x)^{-1}\right)=(-1)(\cosh x)^{-2}\frac{d\cosh x}{dx}\\ &= -\frac{1}{\cosh^2 x}\cdot\sinh x=-\frac{\sinh x}{\cosh^2 x}=-\frac{\sinh x}{\cosh x}\cdot\frac{1}{\cosh x}=-\tanh x\,\text{sech}x.\end{aligned}$$

Example: Find $\frac{d\ \text{csch}x}{dx}$.

$$\begin{aligned}\frac{d\ \text{csch}x}{dx} &= \frac{d}{dx}\left(\frac{1}{\sinh x}\right)=\frac{d}{dx}\left((\sinh x)^{-1}\right)=(-1)(\sinh x)^{-2}\frac{d\sinh x}{dx}\\ &= -\frac{1}{\sinh^2 x}(\cosh x)=-\frac{\cosh x}{\sinh^2 x}=-\frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x}=-\ \coth x\,\text{csch}x.\end{aligned}$$

Example: Find $\frac{d\ \coth x}{dx}$.

$$\begin{aligned}\frac{d\ \coth x}{dx} &= \frac{d}{dx}\left(\frac{1}{\tanh x}\right)=\frac{d(\tanh x)^{-1}}{dx}=(-1)(\tanh x)^{-2}\frac{d\tanh x}{dx}\\ &= -\frac{1}{\tanh^2 x}\frac{d\tanh x}{dx}=-\frac{1}{\tanh^2 x}\cdot\text{sech}^2 x\\ &= -\frac{1}{\frac{\sinh^2 x}{\cosh^2 x}}\cdot\frac{1}{\cosh^2 x}=-\frac{1}{\sinh^2 x}=-\text{csch}^2 x.\end{aligned}$$

10 Derivatives of inverse functions

Example: Explain why $\frac{d\ln x}{dx} = \frac{1}{x}$.

Since $e^{\ln x} = x$, $\quad \frac{de^{\ln x}}{dx} = \frac{dx}{dx}$.

So $\quad e^{\ln x}\frac{d\ln x}{dx} = 1$.

So $\quad x\frac{d\ln x}{dx} = 1$.

So $\quad \frac{d\ln x}{dx} = \frac{1}{x}$.

Example: Find $\frac{d\sin^{-1}x}{dx}$.

Since $\sin(\sin^{-1}x) = x$, $\quad \frac{d\sin(\sin^{-1}x)}{dx} = \frac{dx}{dx}$.

So $\quad \cos(\sin^{-1}x)\frac{d\sin^{-1}x}{dx} = 1$.

So $\quad \frac{d\sin^{-1}x}{dx} = \frac{1}{\cos(\sin^{-1}x)}$.

So we would like to "simplify" $\cos(\sin^{-1}x)$.

Since $1-\cos^2(\sin^{-1}x) = \sin^2(\sin^{-1}x)$, $\quad 1-\big(\cos(\sin^{-1}x)\big)^2 = \big(\sin(\sin^{-1}x)\big)^2$.

So $\quad 1-\big(\cos(\sin^{-1}x)\big)^2 = x^2$. $\quad$ So $\quad 1-x^2 = \big(\cos(\sin^{-1}x)\big)^2$. $\quad$ So $\quad \cos(\sin^{-1}x) = \sqrt{1-x^2}$.

So $\quad \frac{d\sin^{-1}x}{dx} = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$.

Example: Find $\frac{d\cos^{-1}x}{dx}$.

Since $\cos(\cos^{-1}x) = x$, $\quad \frac{d\cos(\cos^{-1}x)}{dx} = \frac{dx}{dx}$.

So $\quad -\sin(\cos^{-1}x)\frac{d\cos^{-1}x}{dx} = 1$. $\quad$ So $\quad \frac{d\cos^{-1}x}{dx} = \frac{-1}{\sin(\cos^{-1}x)}$.

So we would like to "simplify" $\sin(\cos^{-1}x)$.

Since $1-\sin^2(\cos^{-1}x) = \cos^2(\cos^{-1}x)$, $\quad 1-\big(\sin(\cos^{-1}x)\big)^2 = \big(\cos(\cos^{-1}x)\big)^2$.

So $\quad 1-\big(\sin(\cos^{-1}x)\big)^2 = x^2$. $\quad$ So $\quad 1-x^2 = \big(\sin(\cos^{-1}x)\big)^2$.

So $\quad \sin(\cos^{-1}x) = \sqrt{1-x^2}$. $\quad$ So $\quad \frac{d\cos^{-1}x}{dx} = \frac{-1}{\sin(\cos^{-1}x)} = \frac{-1}{\sqrt{1-x^2}}$.

Example: Find $\frac{d\tan^{-1}x}{dx}$.

Since $\tan(\tan^{-1}x) = x$, $\quad \frac{d\tan(\tan^{-1}x)}{dx} = \frac{dx}{dx}$.

So $\quad \sec^2(\tan^{-1} x)\, \dfrac{d \tan^{-1} x}{dx} = 1.$ $\qquad$ So $\quad \dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{\sec^2(\tan^{-1} x)}.$

So we would like to "simplify" $\sec^2(\tan^{-1} x)$.

Since $\quad \sin^2 x + \cos^2 x = 1,$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

So $\quad \tan^2 x + 1 = \sec^2 x.$

So $\quad \sec^2(\tan^{-1} x) = \tan^2(\tan^{-1} x) + 1 = \big(\tan(\tan^{-1} x)\big)^2 + 1 = x^2 + 1.$

So $\quad \dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{x^2+1}.$

Example: Find $\dfrac{d \cot^{-1} x}{dx}$. Since $\quad \cot(\cot^{-1} x) = x, \qquad \dfrac{d \cot(\cot^{-1} x)}{dx} = \dfrac{dx}{dx}.$

So $\quad -\csc^2(\cot^{-1} x)\, \dfrac{d \cot^{-1} x}{dx} = 1.$ $\qquad$ So $\quad \dfrac{d \cot^{-1} x}{dx} = \dfrac{-1}{\csc^2(\cot^{-1} x)}.$

So we would like to "simplify" $\csc^2(\cot^{-1} x)$.

Since $\quad \sin^2 x + \cos^2 x = 1,$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $\quad 1 + \cot^2 x = \csc^2 x.$

So $\quad \csc^2(\cot^{-1} x) = 1 + \cot^2(\cot^{-1} x) = 1 + \big(\cot(\cot^{-1} x)\big)^2 = 1 + x^2.$

So $\quad \dfrac{d \cot^{-1} x}{dx} = \dfrac{-1}{1+x^2}.$

Example: Find $\dfrac{d \sec^{-1} x}{dx}$.

Since $\quad \sec(\sec^{-1} x) = x, \qquad \dfrac{d \sec(\sec^{-1} x)}{dx} = \dfrac{dx}{dx}.$

So $\quad \tan(\sec^{-1} x) \sec(\sec^{-1} x)\, \dfrac{d \sec^{-1} x}{dx} = 1.$ $\qquad$ So $\quad \tan(\sec^{-1} x) \cdot x \cdot \dfrac{d \sec^{-1} x}{dx} = 1.$

So $\quad \dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{x \tan(\sec^{-1} x)}.$

So we would like to "simplify" $\tan(\sec^{-1} x)$.

Since $\quad \sin^2 x + \cos^2 x = 1,$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

So $\quad \tan^2 x + 1 = \sec^2 x.$

So $\quad \tan^2(\sec^{-1} x) + 1 = \sec^2(\sec^{-1} x).$ $\qquad$ So $\quad \big(\tan(\sec^{-1} x)\big)^2 + 1 = \big(\sec(\sec^{-1} x)\big)^2.$

So $\quad \big(\tan(\sec^{-1} x)\big)^2 + 1 = x^2.$ $\qquad$ So $\quad \tan(\sec^{-1} x) = \sqrt{x^2 - 1}.$

So $\quad \dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{x\sqrt{x^2-1}}.$

Example: Find $\dfrac{d \csc^{-1} x}{dx}$.

Since $\csc(\csc^{-1} x) = x$, $\quad \dfrac{d\csc(\csc^{-1} x)}{dx} = \dfrac{dx}{dx}$.

So $\quad -\csc(\csc^{-1} x)\cot(\csc^{-1} x)\dfrac{d\csc^{-1} x}{dx} = 1.$ $\qquad$ So $\quad -x\cot(\csc^{-1} x)\dfrac{d\csc^{-1} x}{dx} = 1.$

So $\quad \dfrac{d\csc^{-1} x}{dx} = \dfrac{-1}{x\cot(\csc^{-1} x)}$.

So we would like to "simplify" $\cot(\csc^{-1} x)$.

Since $\quad \sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $\quad 1 + \cot^2 x = \csc^2 x$.

So $\quad 1 + \cot^2(\csc^{-1} x) = \csc^2(\csc^{-1} x)$. $\qquad$ So $\quad 1 + \big(\cot(\csc^{-1} x)\big)^2 = \big(\csc(\csc^{-1} x)\big)^2$.

So $\quad 1 + \big(\cot(\csc^{-1} x)\big)^2 = x^2$. $\qquad$ So $\quad \cot(\csc^{-1} x) = \sqrt{x^2 - 1}$.

So $\quad \dfrac{d\csc^{-1} x}{dx} = \dfrac{-1}{x\sqrt{x^2-1}}$.

Example: Find $\dfrac{dy}{dx}$ when $y = \log_x 10$.

$$x^y = x^{\log_x 10} = 10.$$

Take the derivative:

$$\frac{d\ x^y}{dx} = \frac{d\ (e^{\ln x})^y}{dx} = \frac{d\ e^{y\ln x}}{dx} = e^{y\ln x}\left(y\cdot\frac{1}{x} + \frac{dy}{dx}\ln x\right)$$
$$= \frac{d10}{dx} = 0.$$

So $\quad e^{y\ln x}\left(y\cdot\dfrac{1}{x} + \dfrac{dy}{dx}\ln x\right) = 0.$

Solve for $\dfrac{dy}{dx}$.

$$e^{y\ln x}\frac{dy}{dx}\ln x = \frac{-e^{y\ln x}y}{x}. \qquad \text{So} \qquad \frac{dy}{dx} = \frac{-e^{y\ln x}y}{xe^{y\ln x}\ln x} = \frac{-y}{x\ln x} = \frac{\log_x 10}{x\ln x}.$$

Example: Find the third derivative of 2^x with respect to x.

$y = 2^x$.

$$\frac{dy}{dx} = \frac{d2^x}{dx} = \frac{2(e^{\ln 2})^x}{dx} = \frac{de^{x\ln 2}}{dx} = e^{x\ln 2}(\ln 2) = (e^{\ln 2})^x\ln 2 = 2^x\ln 2.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d2^x\ln 2}{dx} = \ln 2\cdot 2^x\ln 2 = (\ln 2)^2 2^x.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\big(\ln 2)^2 2^x\big) = (\ln 2)^2 2^x\ln 2 = (\ln 2)^3 2^x.$$

Example: If $y = a\cos(\ln x) + b\sin(\ln x)$ show that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

$$\begin{aligned}
\frac{dy}{dx} &= a(-\sin(\ln x))\frac{1}{x} + b\cos(\ln x)\frac{1}{x} \\
&= -a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1},
\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -a\cos(\ln x)\frac{1}{x}x^{-1} + -a\sin(\ln x)(-1)x^{-2} + \; -b\sin(\ln x)\frac{1}{x}x^{-1} + b\cos(\ln x)(-1)x^{-2} \\
&= \frac{-a\cos(\ln x) + a\sin(\ln x) - b\sin(\ln x) - b\cos(\ln x)}{x^2} \\
&= \frac{1}{x^2}\big((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\big).
\end{aligned}$$

So

$$\begin{aligned}
LHS &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y \\
&= x^2\,\frac{1}{x^2}\big((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\big) \\
&\qquad + x\big(-a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1}\big) \\
&\qquad + a\cos(\ln x) + b\sin(\ln x) \\
&= (a-b)\sin(\ln x) - (a+b)\cos(\ln x) \\
&\quad - a\sin(\ln x) + b\cos(\ln x) \\
&\quad + b\sin(\ln x) + a\cos(\ln x) \\
&= 0.
\end{aligned}$$

Example: Find $\dfrac{dy}{dx}$ when $a\sin(xy) + b\cos\left(\frac{x}{y}\right) = 0$.

Take the derivative:

$$\begin{aligned}
0 &= a\cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) + \; -b\sin\left(\frac{x}{y}\right)\left(x(-1)y^{-2}\frac{dy}{dx} + 1\cdot y^{-1}\right) \\
&= a\cos(xy)x\frac{dy}{dx} + a\cos(xy)y + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} - b\sin\left(\frac{x}{y}\right)y^{-1}.
\end{aligned}$$

Solve for $\dfrac{dy}{dx}$.

$$a\cos(xy)x\frac{dy}{dx} + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} = a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}.$$

So

$$\frac{dy}{dx} = \frac{a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}}{a\cos(xy)x + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}}$$

$$= \frac{a\cos(xy)y^3 - b\sin\left(\frac{x}{y}\right)y}{a\cos(xy)xy^2 + b\sin\left(\frac{x}{y}\right)x}$$

Example: Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{a}{x}\right) \cdot \cot^{-1}\left(\frac{x}{a}\right)$.

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{a}{x}\right)\left(\frac{-1}{1+\left(\frac{x}{a}\right)^2}\right)\frac{1}{a} + \frac{1}{1+\left(\frac{x}{a}\right)^2}(-1)ax^{-2}\cot^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)}{a+\frac{x^2}{a}} + -\cot^{-1}\left(\frac{x}{a}\right)ax^2 + a^2$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)a}{a^2+x^2} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \cot^{-1}\left(\frac{x}{a}\right)\right).$$

If $\frac{a}{x} = \tan z$ then $\frac{x}{a} = \cot z$ and $z = \tan^{-1}\left(\frac{a}{x}\right) = \cot^{-1}\left(\frac{x}{a}\right)$.

So

$$\frac{dy}{dx} = \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{a}{x}\right)\right) = \frac{-2a\tan^{-1}\left(\frac{a}{x}\right)}{a^2+x^2}.$$

11 Taylor's theorem and the limit formula for the deriative

The derivative of f with respect to x is $\frac{df}{dx}$. It is common to write $f'(x)$ in place of $\frac{df}{dx}$.

$$f'(x) = \frac{df}{dx}.$$

The *second derivative of f with respect to x* is

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right),$$

the derivative of the derivative of f. Both $\frac{d^2f}{dx^2}$ and $f''(x)$ are notations for the same thing, the second derivative of f.

The *third derivative of f with respect to x* is

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right),$$

the derivative of the second derivative of f. Use the notations $\frac{d^3f}{dx^3}$ and $f'''(x)$ interchangably for the third derivative of f.

The *fourth derivative of f with respect to x* is

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{d}{dx}\left(\frac{d^3f}{dx^3}\right),$$

the derivative of the third derivative of f.

Let a be a number. Then f *evaluated at a* is

$$f(a) = f\big|_{x=a} = c_0 + c_1a + c_2a^2 + c_3x^3 + \cdots,$$

if $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. Use both notations, $f(a)$ and $f\big|_{x=a}$, interchangably, for f evaluated at a.

Example: If $f(x) = 7x^3 + 3x^2 + 5x + 12$ and $a = 3$ then

$$\begin{aligned} f(3) &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5, \\ f\big|_{x=3} &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5. \end{aligned}$$

$$\begin{aligned} \frac{df}{dx} &= 21x^2 + 6x + 5, & \frac{df}{dx}\Big|_{x=3} &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \\ f' &= 21x^2 + 6x + 5, & f'(3) &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \end{aligned}$$

$$\begin{aligned} \frac{d^2f}{dx^2} &= 42x + 6, & \frac{d^2f}{dx^2}\Big|_{x=3} &= 42\cdot 3 + 6 = 132, \\ f'' &= 42x + 6, & f''(3) &= 42\cdot 3 + 6 = 132, \end{aligned}$$

$$\begin{aligned} \frac{d^3f}{dx^3} &= 42, & \frac{d^3f}{dx^3}\Big|_{x=3} &= 42, \\ f''' &= 42, & f'''(3) &= 42, \end{aligned}$$

$$\begin{aligned} \frac{d^4f}{dx^4} &= 0, & \frac{d^4f}{dx^4}\Big|_{x=3} &= 0, \\ f^{(4)} &= 0, & f^{(4)}(3) &= 0. \end{aligned}$$

Taylor's and Maclaurin's theorems and the limit formula for the derivative

If $\quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots$

then

$$f(a) = c_0,$$

$$\frac{df}{dx}\Big|_{x=a} = \big(c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots\big)\Big|_{x=a} \;=\; c_1,$$

$$\frac{d^2 f}{dx^2}\Big|_{x=a} = \big(2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots\big)\Big|_{x=a} \;=\; 2c_2,$$

$$\frac{d^3 f}{dx^3}\Big|_{x=a} = \big(3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + 6\cdot 5\cdot 4c_6(x-a)^3 + \cdots\big)\Big|_{x=a} \;=\; 3\cdot 2c_3,$$

$$\frac{d^4 f}{dx^4}\Big|_{x=a} = \big(4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + 6\cdot 5\cdot 4\cdot 3c_4(x-a)^2 + \cdots\big)\Big|_{x=a} \;=\; 4\cdot 3\cdot 2c_4,$$

and we can continue this process to find

$$\frac{d^k f}{dx^k}\Big|_{x=a} = k!c_k, \qquad \text{for } k = 1, 2, 3, \ldots.$$

Dividing both sides by $k!$ gives

$$c_k = \frac{1}{k!}\left(\frac{d^k f}{dx^k}\Big|_{x=a}\right).$$

So

$$f(x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \frac{1}{3!}\left(\frac{d^3 f}{dx^3}\Big|_{x=a}\right)(x-a)^3 + \cdots,$$

or, equivalently,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Now subtract $f(a)$ from both sides:

$$f(x) - f(a) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Divide both sides by $x - a$.

$$\frac{f(x) - f(a)}{x-a} = f'(a) + \frac{1}{2!}f''(a)(x-a) + \frac{1}{3!}f'''(a)(x-a)^2 + \frac{1}{4!}f^{(4)}(a)(x-a)^3 + \cdots.$$

Evaluate both sides at $x = a$.

$$\frac{f(x) - f(a)}{x-a}\Big|_{x=a} = f'(a) + 0 + 0 + 0 + 0 + \cdots.$$

So $\quad f'(a) = \left.\frac{f(x)-f(a)}{x-a}\right|_{x=a}.$

Let $x = a+h$. Then $\quad f'(a) = \left.\frac{f(a+h)-f(a)}{a+h-a}\right|_{a+h=a}.$

So $\quad \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{f(a+h)-f(a)}{h}\right|_{h=0}.$

Another way to write this is

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "6".
You ask me what $f'(0)$ is and I say "10".
You ask me what $f''(0)$ is and I say "31".
You ask me what $f'''(0)$ is and I say "5".
You ask me what $f^{(4)}(0)$ is and I say "7".
You ask me what $f^{(5)}(0)$ is and I say "0".
You ask me what $f^{(6)}(0)$ is and I say "0".
You ask me what $f^{(7)}(0)$ is and I say "0, they are all coming out to 0 now.".

At this point you win because you know that

$$\begin{aligned}
f(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 + \cdots \\
&= 6 + 10(x-0) + \frac{1}{2!}31(x-0)^2 + \frac{1}{3!}5(x-0)^3 + \frac{1}{4!}7(x-0)^4 \\
&\qquad + \frac{1}{5!}\cdot 0(x-0)^5 + \frac{1}{6!}\cdot 0(x-0)^6 + \frac{1}{7!}\cdot 0(x-0)^7 + 0 + 0 + \cdots \\
&= 6 + 10x + \frac{31}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{24}x^4,
\end{aligned}$$

and so you have found out what f is.

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "I won't tell you, but $f(3) = 4$".

You ask me what $f'(0)$ is and I say "I won't tell you, but $\left.\frac{df}{dx}\right|_{x=3} = 2$".

You ask me what $f''(0)$ is and I say "I won't tell you, but $\left.\frac{d^2f}{dx^2}\right|_{x=3} = 5$".

You ask me what $f'''(0)$ is and I say "I won't tell you, but $\left.\frac{d^3f}{dx^3}\right|_{x=3} = 0$ and all the rest of the $\left.\frac{d^kf}{dx^k}\right|_{x=3}$ are coming out to 0".

At this point you win because you know that

$$
\begin{aligned}
f(x) &= f\big|_{x=3} + \left(\frac{df}{dx}\Big|_{x=3}\right)(x-3) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=3}\right)(x-3)^2 + \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=3}\right)(x-3)^3 + \cdots \\
&= 2 + 5(x-3) + \frac{1}{2!}5(x-3)^2 + \frac{1}{3!}\cdot 0(x-3)^3 + 0 + 0 + \cdots \\
&= 2 + 5x - 15 + \frac{5}{2}(x^2 - 6x + 9) + 0 + 0 + \cdots \\
&= -13 + 5x + \frac{5}{2}x^2 - 15x + \frac{45}{2} \\
&= \frac{5}{2}x^2 - 10x + \frac{19}{2},
\end{aligned}
$$

and so you know what f is.

12 Limits

Write

$$\lim_{x\to 2} f(x) = 10$$

if $f(x)$ gets closer and closer to 10 as x gets closer and closer to 2.

Example: Evaluate $\lim_{x\to 2} \frac{3x^2+8}{x^2-x}$.

When $x = 1.5$, $\quad \frac{3x^2+8}{x^2-x} = 19.66....$

When $x = 1.9$, $\quad \frac{3x^2+8}{x^2-x} = 11.011....$

When $x = 1.99$, $\quad \frac{3x^2+8}{x^2-x} = 10.091....$

When $x = 1.999$, $\quad \frac{3x^2+8}{x^2-x} = 10.00901....$

When $x = 1.9999$, $\quad \frac{3x^2+8}{x^2-x} = 10.0009001....$

So $\lim_{x\to 2} \frac{3x^2+8}{x^2-x} = 10.$

Usually determining the limit is straightforward.

Example: $\lim_{x\to 1} 6x^2 - 4x + 3 = 5.$

But sometimes ...

Example: $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \stackrel{?}{=} \frac{0}{0}.$

$$\frac{0}{0} \quad \text{MAKES NO SENSE.}$$

Example: $\lim_{x\to 0} \frac{5x}{x} \stackrel{?}{=} \frac{0}{0}.$

$$\lim_{x\to 0}\frac{5x}{x} = \lim_{x\to 0} 5 = 5.$$

Example: $\lim_{x\to 0}\frac{17x}{2x} \overset{?}{=} \frac{0}{0}$.

$$\lim_{x\to 0}\frac{17x}{2x} = \lim_{x\to 0}\frac{17}{2} = \frac{17}{2}.$$

Let's go back to

Example: $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} \overset{?}{=} \frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\cdot\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}\\ &= \lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.\end{aligned}$$

So, whenever a limit *looks* like it is coming out to $\frac{0}{0}$ it needs to be looked at in a different way to see what it is *really* getting closer and closer to.

Example: Evaluate $\lim_{x\to 7}\frac{x^2-49}{x-7}$.

$$\lim_{x\to 7}\frac{x^2-49}{x-7} = \lim_{x\to 7}\frac{(x-7)(x+7)}{x-7} = \lim_{x\to 7}(x+7) = 7+7 = 14.$$

Example: Evaluate $\lim_{x\to 5}\frac{x^5-3125}{x-5}$.

$$\begin{aligned}\lim_{x\to 5}\frac{x^5-3125}{x-5} &= \lim_{x\to 5}\frac{x^5-5^5}{x-5} = \lim_{x\to 5}\frac{(x-5)(x^4+5x^3+5^2x^2+5^3x+5^4)}{x-5}\\ &= \lim_{x\to 5} x^4+5x^3+5^2x^2+5^3x+5^4 = 5^4+5^4+5^4+5^4 = 5^5 = 3125.\end{aligned}$$

Example: Evaluate $\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{x-a}$.

$$\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{x-a}\lim_{x\to a}\frac{(x^{5/2}-a^{5/2})}{(x-a)}\frac{(x^{5/2}+a^{5/2})}{(x^{5/2}+a^{5/2})} = \lim_{x\to a}\frac{x^5-a^5}{x-a}\cdot\frac{1}{x^{5/2}-a^{5/2}}$$

$$= \lim_{x\to a}\frac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a}\cdot\frac{1}{x^{5/2}+a^{5/2}} = \lim_{x\to a}\frac{x^4+ax^3+a^2x^2+a^3x+a^4}{x^{5/2}+a^{5/2}}$$

$$= \frac{a^4+a^4+a^4+a^4+a^4}{a^{5/2}+a^{5/2}} = \frac{5a^4}{2a^{5/2}} = \frac{5}{2}a^{3/2}.$$

Particularly useful limits

Example: Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}{x}$$
$$= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots = 1 - 0 + 0 - 0 + 0 - \cdots = 1.$$

Example: Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x}$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) - 1}{x}$$
$$= \lim_{x\to 0} \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}{x}$$
$$= \lim_{x\to 0} -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \cdots = -0 + 0 - 0 + 0 - \cdots = 0.$$

Example: Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1}{x}$$
$$= \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x}$$
$$= \lim_{x\to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots = 1 + 0 + 0 + 0 + 0 + \cdots = 1.$$

Example: Evaluate $\lim_{x\to 0} \frac{\ln(1+x)}{x}$.

Let $y = \ln(1+x)$. Then

$$e^y = 1 + x, \quad x = e^y - 1, \qquad \text{and} \quad y \to 0 \text{ as } x \to 0.$$

So

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{y\to 0} \frac{y}{e^y - 1} = \lim_{y\to 0} \frac{1}{\frac{e^y - 1}{y}} = \frac{1}{1} = 1.$$

Example: Evaluate $\lim_{x\to 0} (1+x)^{1/x}$.

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} \left(e^{\ln(1+x)}\right)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}} = e^1 = e.$$

Note: $n \to \infty$ means as n gets larger and larger.

Example: Evaluate $\lim_{n\to\infty} (1 + \frac{1}{n})^n$.

Let $x = \frac{1}{n}$. Then $x \to 0$ as $n \to \infty$. So

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{1/x} = e.$$

Example: Evaluate $\lim_{x\to\pi}\frac{\sin x}{x-\pi}$.

Let $y = x - \pi$. Then $y \to 0$ as $x \to \pi$. So

$$\begin{aligned}\lim_{x\to\pi}\frac{\sin x}{x-\pi} &= \lim_{y\to 0}\frac{\sin(y+\pi)}{y} = \lim_{y\to 0}\frac{\sin y\cos\pi + \cos y\sin\pi}{y}\\ &= \lim_{y\to 0}\frac{\sin y(-1)+\cos y\cdot 0}{y} = \lim_{y\to 0}\frac{-\sin y}{y} = -1.\end{aligned}$$

Example: Evaluate $\lim_{x\to\infty}\frac{x^2-7x+11}{3x^2+10}$

$$\lim_{x\to\infty}\frac{x^2-7x+11}{3x^2+10} = \lim_{x\to\infty}\frac{1-\frac{7}{x}+11x^2}{3+\frac{10}{x^2}} = \frac{1-0+0}{3+0} = \frac{1}{3}.$$

Example: Evaluate $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} &= \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 3x\cdot\frac{5x}{\sin 5x}\cdot\frac{1}{5x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\frac{1}{\frac{\sin 5x}{5x}}\frac{3x}{5x}\\ &= \lim_{x\to 0}\frac{\sin 3x}{3x}\frac{1}{\frac{\sin 5x}{5x}}\frac{3}{5} = 1\cdot\frac{1}{1}\cdot\frac{3}{5} = \frac{3}{5}.\end{aligned}$$

Example: Evaluate $\lim_{x\to 1}\frac{1-x}{(\cos^{-1}x)^2}$ Let $y = \cos^{-1}x$. Then $y \to 0$ as $x \to 1$ and $x = \cos y$. So

$$\begin{aligned}\lim_{x\to 1}\frac{1-x}{(\cos^{-1}x)^2} &= \lim_{y\to 0}\frac{1-\cos y}{y^2} = \lim_{y\to 0}\frac{(1-\cos y)}{y^2}\cdot\frac{(1+\cos y)}{(1+\cos y)}\\ &= \lim_{y\to 0}\frac{(1-\cos^2 y)}{y^2}\cdot\frac{1}{1+\cos y} = \lim_{y\to 0}\frac{\sin y}{y}\cdot\frac{\sin y}{y}\cdot\frac{1}{1+\cos y} = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Example: Evaluate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x) = \sin 2x$.

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\sin(2(x+\Delta x))-\sin 2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(2x+2\Delta x)-\sin 2x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sin 2x\cos 2\Delta x+\cos 2x\sin 2\Delta x-\sin 2x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin 2x\cdot\frac{(\cos 2\Delta x-1)}{\Delta x}+\cos 2x\frac{\sin 2\Delta x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin 2x\cdot\frac{(\cos 2\Delta x-1)}{2\Delta x}\cdot 2+\cos 2x\frac{\sin 2\Delta x}{2\Delta x}\cdot 2\\ &= \sin 2x\cdot 0\cdot 2+\cos 2x\cdot 1\cdot 2 = 2\cos 2x.\end{aligned}$$

Example: Evaluate $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\cos x^2$.

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)^2-\cos x^2}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\cos(x^2+2x\Delta x+(\Delta x)^2)-\cos x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\cos x^2\cos(2x\Delta x+(\Delta x)^2)-\sin x^2\sin(2x\Delta x+(\Delta x)^2)-\cos x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\cos x^2\cdot\frac{(\cos(2x\Delta x+(\Delta x)^2)-1)}{\Delta x}-\sin x^2\cdot\frac{\sin(2x\Delta x+(\Delta x)^2)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\cos x^2\frac{(\cos(2x\Delta x+(\Delta x)^2)-1)}{2x\Delta x+(\Delta x)^2}\cdot\frac{2x\Delta x+(\Delta x)^2}{\Delta x}\\
&\qquad\qquad -\sin x^2\frac{\sin(2x\Delta x+(\Delta x)^2)}{2x\Delta x+(\Delta x)^2}\cdot\frac{(2x\Delta x+(\Delta x)^2)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\cos x^2\cdot\frac{(\cos(\text{STUFF})-1)}{\text{STUFF}}\cdot(2x+\Delta x)-\sin x^2\frac{\sin(\text{STUFF})}{\text{STUFF}}\cdot(2x+\Delta x)\\
&= \cos x^2\cdot 0\cdot 2x-\sin x^2\cdot 1\cdot 2x=-2x\sin x^2.
\end{aligned}$$

Example: Evaluate $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=x^x$.

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{(x+\Delta x)^{x+\Delta x}-x^x}{\Delta x} &= \lim_{\Delta x\to 0}\frac{(e^{\ln(x+\Delta x)})^{x+\Delta x}-(e^{\ln x})^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)\ln(x+\Delta x)}-e^{x\ln x}}{\Delta x}\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\cdot\frac{\left(e^{(x+\Delta x)\ln(x+\Delta x)-x\ln x}-1\right)}{\Delta x}\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\frac{(e^{(x+\Delta x)\ln(x+\Delta x)-x\ln x}-1)}{\big((x+\Delta x)\ln(x+\Delta x)-x\ln x\big)}\cdot\frac{\big((x+\Delta x)\ln(x+\Delta x)-x\ln x\big)}{\Delta x}\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\left(\frac{e^{\text{STUFF}}-1}{\text{STUFF}}\right)\left(\frac{x\ln(x+\Delta x)-x\ln x}{\Delta x}+\ln(x+\Delta x)\right)\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\left(\frac{e^{\text{STUFF}}-1}{\text{STUFF}}\right)\left(\frac{x\ln\big(x(1+\frac{\Delta x}{x})\big)-x\ln x}{\Delta x}+\ln(x+\Delta x)\right)\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\left(\frac{e^{\text{STUFF}}-1}{\text{STUFF}}\right)\left(\frac{x\big(\ln x+\ln(1+\frac{\Delta x}{x})\big)-x\ln x}{\Delta x}+\ln(x+\Delta x)\right)\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\left(\frac{e^{\text{STUFF}}-1}{\text{STUFF}}\right)\left(\frac{x\ln(1+\frac{\Delta x}{x})}{\Delta x}+\ln(x+\Delta x)\right)\\
&= \lim_{\Delta x\to 0}e^{x\ln x}\left(\frac{e^{\text{STUFF}}-1}{\text{STUFF}}\right)\left(\frac{\ln(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}}+\ln(x+\Delta x)\right)\\
&= e^{x\ln x}\cdot 1(1+\ln x)=x^x+x^x\ln x.
\end{aligned}$$

13 Graphing and Existence of limits

Example: What is $\lim_{x\to 0}\frac{1}{x}$?

If $x = .1$ then $\frac{1}{x} = 10$.
If $x = .01$ then $\frac{1}{x} = 100$.
If $x = .001$ then $\frac{1}{x} = 1000$.
If $x = .0001$ then $\frac{1}{x} = 10000$.
So, it looks like $\lim_{x\to 0} \frac{1}{x} = \infty$.
If $x = -.1$ then $\frac{1}{x} = -10$.
If $x = -.01$ then $\frac{1}{x} = -100$.
If $x = -.001$ then $\frac{1}{x} = -1000$.
So, it looks like $\lim_{x\to 0} \frac{1}{x} = -\infty$.
Since $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$,

$$\lim_{x\to 0} = \text{UNDEFINED} \qquad \text{PICTURE.}$$

Example: Evaluate $\lim_{x\to -1} \ln x$.

Look at the graph of $\ln x$.

GRAPH of e^x GRAPH of $\ln x$

Notes:
$e^0 = 1, \quad e^1 = 2.718...$
$e^2 \simeq 8.8, \quad e^3 \simeq 25$
$e^{-1} \simeq \frac{1}{3}, \quad e^{-2} \simeq \frac{1}{4}$

Notes:
$y = \ln x$ means $e^y = x$.
So this graph is the same as $y = e^x$
but with x and y swtiched.

So, from the graph, $y = \ln x$ doesn't even make sense for x close to 1. So

$\lim_{x\to -1} \ln x$ is certainly undefined.

Note: If we allow x to get closer and closer to -1 and be a complex number then

$$\ln -1 = i\pi \text{ and } i3\pi \text{ and } i5\pi, \ldots$$

since $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$ so that $\ln -1 = i\pi$. Still

$$\lim_{x\to -1} \ln x \quad \text{is undefined}$$

since it can't be getting closer and closer to $i\pi$, $i3\pi$, $i5\pi$, ..., all at once.

Example: Evaluate $\lim_{x\to\infty} \sin x$.

The graph of $\sin x$ is

GRAPH.

So, as x gets larger and larger, $\sin x$ keeps going back and forth between -1 and $+1$. So $\sin x$ doesn't get closer and closer to anything as x gets larger and larger. So

$$\lim_{x\to\infty} \sin x \quad \text{is undefined.}$$

14 Graphing

14.1 Continuous functions

A function is continuous if it doesn't jump as x changes. The function $f(x)$ is *not* continuous exactly at the places where is jumps.

A function $f(x)$ is continuous at $x = a$ if $f(x)$ doesn't jump at $x = a$.

PICTURE	PICTURE
Continuous at $x = a$	Not continuous at $x = a$

In other words, a function $f(x)$ is continuous at $x = a$ if

$$f(x) \text{ gets closer and closer to } f(a) \qquad \text{as } x \text{ gets closer and closer to } a.$$

In other words, a function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Example: $f(x) = \lfloor x \rfloor$, the *round down function.*

$$\lfloor 3.2 \rfloor = 3 \qquad\qquad PICTURE$$

$f(x) = \lfloor x \rfloor$ is continuous if $x \neq 0, \pm 1, \pm 2, \pm 3, \ldots$.

$$\text{Note:} \qquad \lim_{x \to 1^-} \lfloor x \rfloor = 0 \quad \text{and} \quad \lim_{x \to 1+} \lfloor x \rfloor = 1.$$

The *round up function* is $f(x) = \lceil x \rceil$. For example, $\lceil 3.2 \rceil = 4$.

Example: $f(x) = \begin{cases} 1 + x^2, & 0 \leq x \leq 1, \\ 2 - x, & x > 1. \end{cases}$

$$PICTURE$$

$f(x)$ jumps at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 + x^2 = 2, \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 1.$$

So

$$\lim_{x \to 1} f(x) \ \text{ is UNDEFINED.}$$

Example: $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

$\sin 3x$ is continuous everywhere and x is continuous everywhere so

$$\frac{\sin 3x}{x} \ \text{ is continuous everywhere,} \quad \text{EXCEPT}$$

it makes *no sense* when $x = 0$.

What is happening when $x = 0$?

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3 = 1 \cdot 3 = 3, \qquad BUT \qquad f(0) = 1.$$

So $\lim_{x \to 0} f(x) \neq f(0)$ and $f(x)$ is *not* continuous at $x = 0$.

PICTURE OF $y = \sin x$.

PICTURE OF $y = \sin 3x$.

PICTURE OF $y = \frac{1}{x}$ PICTURE OF $y = \dfrac{\sin 3x}{x}$.

14.2 Graphing: Lecture 14

Graphing techniques.

(a) Basic Graphs

(b) Shifting

(c) Scaling

(d) Flipping

(e) limits

(f) asymptotes

(g) Slopes: Increasing/Decreasing

(h) Concave up/Concave down/points of inflection

Basic Graphs

graph of $y = x$ graph of $y = x^2$

graph of $x^2 + y^2 = 1$ graph of $x^2 - y^2 = 1$

graph of $y = \sin x$

graph of $y = \cos x$

graph of e^x

Shifting

Example: Graph $(x - 3)^2 + (y + 2)^2 = 1$.

graph

Notes:

(a) $x^2 + y^2 = 1$ is a circle of radius 1

(b) center is shifted by
3 to the right in the x-direction
2 upwards in the y-direction

Scaling

Example. Graph $2y = \sin 3x$.

graph

Notes:
(a) $y = \sin x$ is the basic graph
(b) the x-axis is scaled (squished) by 3
(c) the y-axis is scaled by 2

Flipping

Example. Graph $y = -e^{-x}$.

graph

Notes:
(a) $y = e^x$ is the basic graph
(b) $y = -e^{-x}$ is the same as $-y = e^{-x}$.
(c) The x-axis is flipped
(d) The y-axis is flipped

Example. Graph $y = \sin\left(\frac{1}{x}\right)$.

graph

Notes:
(a) $y = \sin x$ is the basic graph
(b) Positive x-axis is flipped
(c) Negative x-axis is flipped
(d) As $x \to \infty$, $\sin\left(\frac{1}{x}\right) \to 0^+$
(e) As $x \to -\infty$, $\sin\left(\frac{1}{x}\right) \to 0^-$
(f) As $x \to 0^+$, $\sin\left(\frac{1}{x}\right)$ goes between 1 and -1

Example. Graph $y = \sin^{-1} x$.

graph

Notes:
(a) $y = \sin x$ is the basic graph
(b) $y = \sin^{-1} x$ is the same as $\sin y = x$ so
the x-axis and y axis are switched from $y = \sin x$.

Asymptotes

A *asymptote* of a graph $y = f(x)$ as $x \to a$ is another graph $y = g(x)$ that the original graph gets closer and closer to as x gets closer and closer to a.

Example. Graph $x^2 - y^2 = 1$.

graph

(a) If $y = 0$ then $x = \pm 1$.
(b) $x^2 - y^2 = 1$ is the same as $1 - \frac{y^2}{x^2} = \frac{1}{x^2}$

As $x \to \infty$, $1 - \frac{y^2}{x^2} = \frac{1}{x^2}$ becomes $1 - \left(\frac{y}{x}\right)^2 = 0$. So, as $x \to \infty$, $y^2 = x^2$. So $y = \pm x$. So $y = x$ is an asymptote as $x \to \infty$ and $y = -x$ is also an asymptote as $x \to \infty$.

Example. Graph $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

Since $f(0) = 1$ and

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}, \qquad \lim_{x \to 0} f(x) \neq f(0).$$

So $f(x)$ is *not* continuous at $x = 0$.

graph of $y = \cos x$

graph of $y = -\cos x$

graph of $y = 1 - \cos x$

graph of $y = 1/x^2$

graph of $y = f(x)$

Notes:
(a) As $x \to 0$, $\frac{1-\cos x}{x^2} \to \frac{1}{2}$
(b) $f(0) = 1$
(c) At the peaks of $1 - \cos x$, $\frac{1-\cos x}{x^2} = \frac{2}{x^2}$

14.3 Graphing: Lecture 15

A function $f(x)$ is *continuous* at $x = a$ if it doesn't jump at $x = a$,

$$\text{i.e. if} \quad \lim_{x \to a} f(x) = f(a).$$

PICTURE
not continuous at $x = a$

Think about

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(x)}{\Delta x}$$

in terms of the graph

PICTURE

$$\begin{aligned} \frac{f(a + \Delta x) - f(x)}{\Delta x} &= \frac{\text{change in } f}{\text{change in } x} \\ &= \frac{\text{rise}}{\text{run}} \\ &= \text{slope of line connecting} \\ &\quad (a, f(x)) \text{ and } (a + \Delta x, f(a + \Delta x)) \end{aligned}$$

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(x)}{\Delta x} = \text{slope of } f \text{ at the point } x = a.$$

A function $f(x)$ is *differentiable* at $x = a$ if the derivative $\left.\frac{df}{dx}\right|_{x=a}$ exists, i.e., if the slope of the graph of $f(x)$ at $x = a$ exists.

Example. Graph $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$

graph

Then

$$\left.\frac{df}{dx}\right|_{x=a} = \begin{cases} 1, & \text{if } a > 0, \\ -1, & \text{if } a < 0, \\ \text{does not exist}, & \text{if } a = 0. \end{cases}$$

So f is not differentiable at $x = 0$.

15 Introduction

Things we need to learn in the first couple of weeks of calculus class:

Why $2^0 = 1$.

What $1/6432$ really means.

What -6342 really means.

What 0 means.

What 2^x means.

What $\sin x$ means

What e^x means

What $16^{1/4}$ means.

How to graph complex numbers.

That $\sqrt{9} = 3$ on Mondays and $\sqrt{9} = -3$ on Tuesdays.

How to derive the quadratic formula.

One of the most difficult things in teaching calculus is to settle on the appropriate space of functions. Whatever this space is, it should include

$$x^2, \quad x^-2, \quad \sqrt{x}, \quad e^x, \quad \ln x, \quad \sin x, \quad , \cos x, \quad \tan x, \quad , \cot x, \quad , \sec x, \quad \csc x$$

and

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

and

$$x^x$$

and be closed under composition of functions. Since many of these functions are multivalued this is a tall order. Whether you admit it or not, this is one of the difficulties you will have to deal with when teaching Calculus.

One of the most difficult things in teaching calculus is to decide on a definition of the real numbers If the decision is to try to avoid defining the real numbers then many of the proofs in a calculus class are incomplete and it is no wonder that our students come away with little desire to "see why something is true".

One must work with the admission that e^x is more fundamental than $\sin x$ and $\cos x$, and that $\sin x$ and $\cos x$ are "children" of e^x.

References

[Ch1] I.V. Cherednik, *Double affine Hecke algebras and difference Fourier transforms*, Invent. Math. **152** (2003), no. 2, 213–303. MR1974888, arXiv:math.QA/0110024.