620-295 Real Analysis with Applications

Assignment 5: Due 5pm on 16 October

Lecturer: Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville VIC 3010 Australia
aram@unimelb.edu.au

Due 5pm on 16 October in the appropriate assignment box on the ground floor of Richard Berry.

1. (a) What is the *Fundamental Theorem of Calculus*?

   (b) Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

   (c) Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

   (d) Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this, compute an area by summing up the areas of tiny boxes and then show that applying the Fundamental Theorem of Calculus gives the same result.

2. Using appropriate tests decided whether the following series converge absolutely, converge conditionally or diverge, giving a brief explanation:

   (i) $\sum_{n=1}^{\infty} \frac{3^n}{n!2^n}$

   (ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{\sqrt{n}}$

   (iii) $\sum_{n=1}^{\infty} \frac{n+4}{2-3n\sqrt{n}}$

3. Find the radius of convergence and the interval of convergence of the following power series: $\sum_{n=1}^{\infty} \frac{(-2)^n(x-2)^n}{n3^n}$.

4. You are given that $\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}\cdots\left(\frac{1}{2}+n-1\right)}{n!} x^n$.

   (i) Find a Maclaurin expansion for $\frac{1}{\sqrt{1+x^2}}$.

(ii) Using your answer to (i) find a Maclaurin expansion for $\operatorname{arcsinh} x$. For what values of x does your series converge?

(iii) Find a degree 3 Taylor polynomial for $\cos x$ around the point $x = \frac{1}{2}$ and give Lagrange's form of the remainder.

5. Decide whether the given sequence converges, and if so, find its limit using standard limits, limit theorems, or by referring to a continuous function.

(i) $a_n = \exp\left(\dfrac{3^n - n^2}{5^n + 7n}\right)$

(ii) $b_n = \dfrac{4 - n^4}{n^3 - 7n^{\frac{1}{n}}}$

(iii) $c_n = \dfrac{\log(n+2)}{\log(n+1)}$

6. (i) Show that $a_n = \left(\dfrac{n+3}{n}\right)^n$ is a bounded sequence.

(ii) Is every bounded sequence convergent? If so, give a proof; if not give a counterexample.

7. A sequence (a_n) is defined by $a_{n+1} = \sqrt{2 + a_n}$ and $a_1 = 3$.

(a) Show that $2 \leq a_n \leq 3$.

(b) Show that (a_n) is monotonically decreasing.

(c) Stating appropriate results prove that (a_n) converges and find its limit.

8. Evaluate the following limits, if they exist:

(i) $\lim\limits_{x \to 0} \dfrac{\cos^2 x - 1}{x^2}$

(ii) $\lim\limits_{x \to \infty} \sqrt{x^2 + 3x} - x$

(c) Using the definition of the limit show that $\lim\limits_{x \to 3} \dfrac{x^2 - 2x - 4}{3 - x^2} = \dfrac{1}{6}$.

9. Classify the following improper integrals and evaluate them if they converge:

(i) $\displaystyle\int_1^5 \frac{4x}{\sqrt{x^2 - 1}}.$

(ii) $\int_1^{\infty} \frac{1}{1+x^2}$.

(iii) Does the following integral diverge or converge? Explain why, but do not evaluate the integral. $\int_1^{\infty} \frac{x^2}{(x-2)(x^{11}+2)^{1/4}}$.

10. (a) Use mathematical induction to prove that for any positive integer n, $\sum_{j=1}^{n} j(3j+1) = n(n+1)^2$.

(b) Give the definition of a Cauchy sequence.

(c) Prove that every convergent sequence is Cauchy.

(2)(i) Using the root test on $\sum_{n=1}^{\infty} \frac{3^n}{n!\,2^n}$,

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!\,2^{n+1}}}{\frac{3^n}{n!\,2^n}} \right| = \lim_{n\to\infty} \frac{3}{(n+1)\,2} = 0$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ and $\sum_{n=1}^{\infty} \frac{3^n}{n!\,2^n}$ converges absolutely.

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{\sqrt{n}}$ will converge (by Leibniz's test)

if there exists N such that $\frac{\log n}{\sqrt{n}}$ is monotone and bounded for $n > N$.

Since $\frac{d}{dx}\left(\frac{\log x}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\log x\right) = \frac{-1}{2} x^{-3/2}\log x + x^{-\frac{1}{2}}x^{-1}$

$$= x^{-3/2}\left(\frac{-1}{2}\log x + 1\right) = \frac{x^{-3/2}}{2}(2 - \log x),$$

the function $\frac{\log x}{\sqrt{x}}$ is decreasing if $x > e^2$,

has a maximum at $x = e^2$, and

is increasing if $0 < x < e^2$.

So $\frac{\log n}{\sqrt{n}} < \frac{\log(e^2)}{\sqrt{e^2}} = \frac{2}{e}$ provided $n > 9 > e^2$.

and is monotone decreasing for $n > 9$.

(2 iii) $$\sum_{n=1}^{\infty} \frac{n+4}{2-3n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1+\frac{4}{n}}{\frac{2}{n}+3\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1+\frac{4}{n}}{-3\sqrt{n}+\frac{2}{n}}$$

This grows like $\frac{-1}{3}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ with diverges since $\frac{1}{2}<1$.

More carefully,

$$\sum_{n=1}^{\infty} \frac{1+\frac{4}{n}}{-3\sqrt{n}+\frac{2}{n}} = \frac{-1}{3}\sum_{n=1}^{\infty} \frac{1+\frac{4}{n}}{\sqrt{n}-\frac{2}{3n}}$$

~~Since~~ Think,

$$\frac{1+\frac{4}{1}}{\sqrt{1}-\frac{2}{3}} = \frac{5}{\frac{1}{3}} = 15, \quad \frac{1+\frac{4}{2}}{\sqrt{2}-\frac{2}{6}} > \frac{3}{2}, \quad \frac{1+\frac{4}{3}}{\sqrt{3}-\frac{2}{9}} > \frac{7}{9}$$

$$\frac{1+\frac{4}{4}}{\sqrt{4}-\frac{2}{12}} > \frac{8}{4\cdot 2} = 1, \quad \frac{1+\frac{4}{5}}{\sqrt{5}-\frac{2}{15}} > \frac{9}{10}.$$

So $\sum_{n=1}^{\infty} \frac{1+4/n}{\sqrt{n}-\frac{2}{3n}} > \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ ~~provided~~ since $\sqrt{n}-\frac{2}{3n} > 0$ for $n>1$.

So $\sum_{n=1}^{\infty} \frac{1+4/n}{\sqrt{n}-\frac{2}{3n}}$ diverges.

So $\frac{-1}{3}\sum_{n=1}^{\infty} \frac{1+4/n}{\sqrt{n}-\frac{2}{3n}}$ diverges.

(3(i)) Find the radius of convergence and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-2)^n (x-2)^n}{n\, 3^n}$.

Think about $\sum_{n=1}^{\infty} \frac{(-2)^n z^n}{n\, 3^n}$.

Using the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1} z^{n+1}}{(n+1)\, 3^{n+1}}}{\frac{(-2)^n z^n}{n\, 3^n}} \right| = \lim_{n\to\infty} \frac{2|z|\, n}{(n+1)\, 3} = \lim_{n\to\infty} \frac{2}{3}|z| \frac{1}{(1+\frac{1}{n})}$$

$$= \frac{2}{3}|z|.$$

So $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ if $|z| < \frac{3}{2}$

So the radius of convergence is $3/2$

and the series converges for $x-2 \in (-\frac{3}{2}, \frac{3}{2})$.

So the series converges for $x \in (2-\frac{3}{2}, 2+3/2)$.

What happens if $x = 2+\frac{3}{2}$?

Then $\sum_{n=1}^{\infty} \frac{(-2)^n (2+\frac{3}{2}-2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-2)^n 3^n}{3^n\, 2^n} = \sum_{n=1}^{\infty} (-1)^n$,

which diverges.

3.2

What happens if $x = 2 - 3/2$?

Then
$$\sum_{n=1}^{\infty} \frac{(-2)^n (2 - \frac{3}{2} - 2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} \frac{3^n}{(-2)^n} = \sum_{n=1}^{\infty} 1$$

which diverges.

So the series converges for $x \in (2 - \frac{3}{2}, 2 + \frac{3}{2})$.

4,1

(4)(i) Since

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\cdots\left(\frac{1}{2}+n-1\right)}{n!}\, x^n$$

then

$$\frac{1}{\sqrt{1+x^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\cdots\left(\frac{1}{2}+n-1\right)}{n!}\, x^{2n}$$

(ii) Since $\operatorname{arcsinh} x = \int \frac{1}{\sqrt{1+x^2}}\,dx$

$$\operatorname{arcsinh} x = \int \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{1}{2}+n-1\right)}{n!}\, x^n\right) dx$$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{1}{2}+n-1\right)}{n!}\, x^{n+1}$$

(iii) $\cos x = \cancel{1 - x + \frac{x^2}{2!} + x^3}\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

So a degree 3 Taylor polynomial for $\cos x$ around $x = \frac{1}{2}$ is

$$\cos\left(\tfrac{1}{2}+x\right) = 1 - \frac{\left(\frac{1}{2}+x\right)^2}{2!} + \frac{\left(\frac{1}{2}+x\right)^4}{4!} - \cdots$$

$$= 1 - \frac{1}{2}\left(\frac{1}{4} + x + x^2\right) + \frac{1}{24}\left(\right.$$

$\cos\frac{1}{2} = \cos\frac{1}{2}$,

$\frac{d\cos x}{dx} = -\sin x$ and $\left.\left(\frac{d}{dx}\cos x\right)\right|_{x=1/2} = -\sin\frac{1}{2}$.

$$\frac{d^2}{dx^2}\cos x = -\cos x \quad \text{and} \quad \left(\frac{d^2}{dx^2}\cos x\right)\Bigg|_{x=\frac{1}{2}} = -\cos\tfrac{1}{2}$$

$$\frac{d^3}{dx^3}\cos x = \sin x \quad \text{and} \quad \left(\frac{d^3}{dx^3}\cos x\right)\Bigg|_{x=\frac{1}{2}} = \sin\tfrac{1}{2}.$$

So a degree 3 Taylor polynomial for $\cos x$ at $x=\frac{1}{2}$ is

$$\cos x = \cos\tfrac{1}{2} - (\sin\tfrac{1}{2})(x-\tfrac{1}{2}) - \tfrac{1}{2!}\cos\tfrac{1}{2}(x-\tfrac{1}{2})^2$$
$$+ \tfrac{1}{3!}\sin\tfrac{1}{2}(x-\tfrac{1}{2})^3 + \tfrac{1}{4!}\cos(c)(x-\tfrac{1}{2})^4.$$

where c is between $\frac{1}{2}$ and x.

Lagrange's form of the remainder is

remainder $= \frac{1}{4!}\cos(c)(x-\frac{1}{2})^4$, where c is between $\frac{1}{2}$ and x.

(5,1)

(5) (i) $a_n = \exp\left(\frac{3^n - n^2}{5^n + 7n}\right)$

Since $\exp x = f(x)$ is continuous, the sequence (a_n) will converge if and only if the sequence (b_n) given by

$$b_n = \frac{3^n - n^2}{5^n + 7n} \quad \text{converges.}$$

Since $\lim_{n\to\infty} \frac{n^2}{5^n} = 0$ ~~then~~ and $\lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$ then

$$\lim_{n\to\infty} \frac{3^n - n^2}{5^n + 7n} = \lim_{n\to\infty} \frac{\left(\frac{3}{5}\right)^n - \frac{n^2}{5^n}}{1 + \frac{7n}{5^n}} = \lim_{n\to\infty} \frac{\left(\frac{3}{5}\right)^n - \frac{n^2}{5^n}}{1 + \frac{7n}{5^n}} = \frac{0 - 0}{1} = 0.$$

(ii) Let $b_n = \frac{4 - n^4}{n^3 - 7n^{\frac{1}{n}}}$

Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ then

$$\lim_{n\to\infty} \frac{4 - n^4}{n^3 - 7n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{\frac{4}{n^3} - n}{1 - 7\frac{n^{\frac{1}{n}}}{n^3}} \text{ grows like } \lim_{n\to\infty} -n$$

So this sequence diverges.

(iii) Let $c_n = \frac{\log(n+2)}{\log(n+1)} = \frac{\log\left(n\left(1+\frac{2}{n}\right)\right)}{\log\left(n\left(1+\frac{1}{n}\right)\right)} = \frac{\log n + \log\left(1+\frac{2}{n}\right)}{\log n + \log\left(1+\frac{1}{n}\right)}.$

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$$\lim_{n\to\infty} \frac{\log(n+2)}{\log(n+1)} = \lim_{n\to\infty} \frac{\log n + \log(1+\frac{2}{n})}{\log n + \log(1+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{1+\frac{\log(1+\frac{2}{n})}{\log n}}{1+\frac{\log(1+\frac{1}{n})}{\log n}} = 1,$$

since $\lim_{n\to\infty} \log(1+\frac{2}{n}) = 0$ and $\lim_{n\to\infty} \frac{1}{\log n} = 0$ so that

$$\lim_{n\to\infty} \frac{\log(1+\frac{2}{n})}{\log n} = 0.$$

(6) (i) $a_n = \left(\frac{n+3}{n}\right)^n = \left(1+\frac{3}{n}\right)^n$.

We recall that $\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$.

Hence we'd guess that a_n is bounded by $3^3 = 27$.

To show: (a_n) is bounded by 27

To show: There exists $b \in \mathbb{R}$ such that if $n \in \mathbb{Z}_{>0}$ then $\left(1+\frac{3}{n}\right)^n < b$.

Let $b = 3^3 = 27$

To show: If $n \in \mathbb{Z}_{>0}$ then $\left(1+\frac{3}{n}\right)^n < 27$

Assume $n \in \mathbb{Z}_{>0}$.

To show: $\left(1+\frac{3}{n}\right)^n < 27$

$$\left(1+\frac{3}{n}\right)^n = \exp\left(\log\left(1+\frac{3}{n}\right)^n\right) = \exp\left(n\log\left(1+\frac{3}{n}\right)\right)$$

$$= \exp\left(\frac{3\log\left(1+\frac{3}{n}\right)}{3/n}\right) = \exp\left(\frac{\log\left(1+3/n\right)}{3/n}\right)^3$$

To show: $\exp\left(\frac{\log\left(1+\frac{3}{n}\right)}{3/n}\right) < 3$.

To show: $\frac{\log\left(1+\frac{3}{n}\right)}{3/n} < \log(3)$ $\quad (\log(3) > \log(e) = 1)$.

$$\left|\frac{\log\left(1+\frac{3}{n}\right)}{3/n}\right| = \left|\frac{\frac{3}{n} - \frac{1}{2}\left(\frac{3}{n}\right)^2 + \frac{1}{3}\left(\frac{3}{n}\right)^3 + \cdots}{3/n}\right|$$

$$= \left|1 - \frac{1}{2}\left(\frac{3}{n}\right) + \frac{1}{3}\left(\frac{3}{n}\right)^2 + \cdots\right| <$$

since $\left|1-\frac{1}{2}\cdot 3\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$ and $\left|1-\frac{1}{2}\cdot\frac{3}{2}\right| = \left|\frac{1}{4}\right| = \frac{1}{4}, \ldots$

(6.2)

(6(ii)) The sequence $a_n = (-1)^n$ is bounded above by 1 and bounded below by -1 but is not convergent,

$(a_n) = (-1, 1, -1, 1, \ldots)$.

(7) (a) $a_1 = 3$ and $a_2 = \sqrt{2+3}$ and $a_{n+1} = \sqrt{2+a_n}$

To show: $2 \le a_n \le 3$.

Case $n=1$: Since $a_1 = 3$, $2 \le a_1 \le 3$.

Case $n=2$, Since $4 \le 5 \le 9$ and $a_2 = \sqrt{5}$, $2 \le a_2 \le 3$.

Induction step: Assume $2 \le a_r \le 3$ for $r < n$.

To show: $2 \le a_n \le 3$.

To show: $2 \le \sqrt{2+a_{n-1}} \le 3$.

We know $4 \le 2 + a_{n-1} \le 5$ since $2 \le a_{n-1} \le 3$

so $4 \le 2 + a_{n-1} \le 9$.

So $2 \le \sqrt{2+a_{n-1}} \le 3$.

So $2 \le a_n \le 3$.

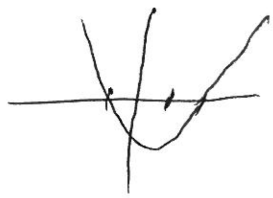
(b) To show: $a_{n+1} \le a_n$.

To show: $\sqrt{2+a_n} \le a_n$

To show: $2 + a_n \le a_n^2$.

To show: $a_n^2 - a_n - 2 \ge 0$.

To show: $(a_n - 2)(a_n + 1) \ge 0$

The graph of $y = x^2 - x - 2$ is 

and $x^2 - x - 2 \ge 0$ if $x \ge 2$.

Since $a_n \ge 2$, by part (a), then $a_n^2 - a_n - 2 \ge 0$

and $a_{n+1} \le a_n$.

(7c) Since (a_n) is monotonically decreasing and bounded below by 2 it will converge.

The limit will be l such that

$$l = \sqrt{2+l}.$$

so $l^2 = 2 + l$ and $l^2 - l - 2 = 0$.

so $l = 2$ or -1 and since the $a_n \geq 2$, $l \geq 2$.

so $l = 2$ and $\lim_{n \to \infty} a_n = 2$.

8.1

8(i) $\lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{x^2} = \lim\limits_{x\to 0} \dfrac{(\cos x - 1)}{x^2}(\cos x + 1)$

$= \frac{-1}{2}(1+1)$, since $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^2} = -\frac{1}{2}$

and $\lim\limits_{x\to 0} \cos x = \cos 0 = 1.$

To recall why $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^2} = -\frac{1}{2}$ do

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots}{x^2} = \lim_{x\to 0} -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots$$

(8 ii) $\lim\limits_{x\to\infty} \sqrt{x^2+3x} - x = \lim\limits_{x\to\infty} (\sqrt{x^2+3x} - x)\dfrac{(\sqrt{x^2+3x} + x)}{\sqrt{x^2+3x} + x}$

$$= \lim_{x\to\infty} \frac{x^2 + 3x - x^2}{\sqrt{x^2+3x} + x} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1 + \frac{3}{x}} + 1} = \frac{3}{\sqrt{1+0} + 1} = \frac{3}{2}.$$

(8c) To show: $\lim\limits_{x \to 3} \dfrac{x^2-2x-4}{3-x^2} = \dfrac{1}{6}$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-3|<\delta$ then $\left|\dfrac{x^2-2x-4}{3-x^2} - \dfrac{1}{6}\right| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $|x-3|<\delta$ then $\left|\dfrac{x^2-2x-4}{3-x^2} - \dfrac{1}{6}\right| < \varepsilon$.

Let $\delta =$

To show: If $|x-3|<\delta$ then $\left|\dfrac{x^2-2x-4}{3-x^2} - \dfrac{1}{6}\right| < \varepsilon$.

Assume $|x-3|<\delta$

To show: $\left|\dfrac{x^2-2x-4}{3-x^2} - \dfrac{1}{6}\right| < \varepsilon$

Since

$$
\begin{aligned}
x^2-2x-4 &= ((x-3)^2+6x-9) - (2(x-3)+6) - 4 \\
&= (x-3)^2 + 6(x-3) + 18 - 9 - 2(x-3) - 6 - 4 \\
&= (x-3)^2 + 4(x-3) - 1
\end{aligned}
$$

and

$$
\begin{aligned}
3-x^2 &= 3 - ((x-3)^2+6x-9) \\
&= 3 - (x-3)^2 - 6(x-3) - 18 + 9 \\
&= -(x-3)^2 - 6(x-3) - 6,
\end{aligned}
$$

8.3

$$\left|\frac{x^2-2x-4}{3-x^2}-\frac{1}{6}\right| = \left|\frac{(x-3)^2+4(x-3)-1}{-(x-3)^2-6(x-3)-6}-\frac{1}{6}\right|$$

$$= \left|\frac{6(x-3)^2+24(x-3)-6+(x-3)^2+6(x-3)+6}{6\left(-(x-3)^2-6(x-3)-6\right)}\right|$$

$$= \left|\frac{7(x-3)^2+30(x-3)}{6\left(-(x-3)^2-6(x-3)-6\right)}\right| = |x-3|\left|\frac{7(x-3)+30}{6\left(-(x-3)^2-6(x-3)-6\right)}\right|$$

$$= \frac{|x-3|}{6}\,\frac{|7(x-3)+30|}{|(x-3)^2+(x-3)+6|} \le \frac{|x-3|}{6}\,\frac{30+7}{6-1-1}$$

$$< \frac{\delta}{6}\cdot\frac{37}{4} = \varepsilon., \quad \text{since } |x-3|<1 \text{ and } |x-3|<\delta$$

$$\text{and } \delta = \frac{24}{37}\varepsilon.$$

19) (i) $\int_1^5 \frac{4x}{\sqrt{x^2-1}}\,dx = \int_1^5 \frac{2\cdot 2x}{(x^2-1)^{\frac{1}{2}}}\,dx$

$= \lim_{b\to 1}\int_b^5 \frac{2\cdot 2x}{(x^2-1)^{\frac{1}{2}}}\,dx = \lim_{b\to 1}\left(2\cdot 2(x^2-1)^{\frac{1}{2}}\Big|_{x=b}^{x=5}\right)$

$= \lim_{b\to 1}\left(4(5^2-1)^{\frac{1}{2}} - 4(b^2-1)^{\frac{1}{2}}\right)$

$= 4\cdot 24^{\frac{1}{2}} = 4\cdot(4\cdot 6)^{\frac{1}{2}} = 8\sqrt{6}.$

This is an improper integral of the 2nd kind.

(ii) $\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{1+x^2}\,dx$

$= \lim_{b\to\infty}\left(\tan^{-1}x\Big|_{x=1}^{x=b}\right) = \lim_{b\to\infty}(\arctan b - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

(ii) $\int_1^\infty \frac{x^2}{(x-2)(x^{11}+2)^{1/4}}\,dx \leq \int_1^\infty \frac{x^2}{(x-2)(x^{11/4})}\,dx$

$= \int_1^\infty \frac{1}{(x-2)\,x^{3/4}}\,dx = \int_1^2 \frac{1}{(x-2)\,x^{3/4}}\,dx + \int_2^\infty \frac{1}{(x-2)\,x^{3/4}}\,dx$

$= \int_1^2 \frac{1}{(x-2)x^{3/4}}\,dx + \int_2^3 \frac{1}{(x-2)x^{3/4}}\,dx + \int_3^\infty \frac{1}{(x-2)x^{3/4}}\,dx.$

9.2

So $\displaystyle\int_1^\infty \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}} = \int_1^2 \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}} + \int_2^3 \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}} + \int_3^\infty \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}}$

Then

$$\int_2^3 \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}} \geq \int_2^3 \frac{2^2}{(x-2)(3^4+2)^{1/4}}\,dx$$

$$= \int_0^1 \frac{4}{(3^4+2)^{1/4}}\,\frac{1}{x}\,dx$$

and

$$\lim_{b\to 0}\int_b^1 \frac{1}{x}\,dx = \cancel{\log(x)}\ \lim_{b\to 0}\left(\log x\Big|_{x=b}^{x=1}\right)$$

$$= \lim_{b\to 0}(\log 1 - \log(b)) \text{ diverges.}$$

So $\displaystyle\int_1^\infty \frac{x^2\,dx}{(x-2)(x^4+2)^{1/4}}$ diverges.

(10) (i) To show: $\sum_{j=1}^{n} j(3j+1) = n(n+1)^2$.

Case $n=1$ $\quad 1\cdot(3\cdot 1+1)=4$ and $1(1+1)^2=4$.

So $\sum_{j=1}^{1} j(3j+1) = 1(1+1)^2$.

Induction step: Assume $\sum_{j=1}^{r} j(3j+1) = r(r+1)^2$ for $r<n$

To show: $\sum_{j=1}^{n} j(3j+1) = n(n+1)^2$.

$$\sum_{j=1}^{n} j(3j+1) = \left(\sum_{j=1}^{n-1} j(3j+1)\right) + n(3n+1)$$

$$= (n-1)(n-1+1)^2 + n(3n+1), \quad \text{by induction,}$$

$$= (n-1)n^2 + 3n^2 + n$$

$$= n((n-1)n + 3n + 1)$$

$$= n(n^2+2n+1) = n(n+1)^2. \quad //$$

(ii) A sequence (a_n) is Cauchy if it satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists ~~$\delta \in \mathbb{R}_{>0}$~~ $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|a_m - a_n| < \varepsilon$.

10.2

(iii) To show: If (a_n) is a sequence and $\lim_{n\to\infty} a_n$ exists then (a_n) is Cauchy.

Assume (a_n) is a sequence and $\lim_{n\to\infty} a_n$ exists.

So there exists l such that $\lim_{n\to\infty} a_n = l$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|a_m - a_n| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

We know: There exists N_1 such that if $n > N_1$ then $|a_n - l| < \varepsilon/2$.

Let $N = N_1$

To show: If $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $|a_m - a_n| < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$.

To show: $|a_m - a_n| < \varepsilon$

$$
\begin{aligned}
|a_m - a_n| &= |(a_m - l) - (a_n - l)| \\
&\leq |a_m - l| + |a_n - l| \\
&< \varepsilon/2 + \varepsilon/2 < \varepsilon.
\end{aligned}
$$

So (a_n) is Cauchy. //