Problem Set for 620-295

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1. Problems

Items marked with **[???]** need attention.

(1)
  a. Define ordered monoid.
  b. Define $\mathbb{Z}_{>0}$.
  c. Show that $\mathbb{Z}_{>0}$ is an ordered monoid.

(2)
  a. Define $\mathbb{Z}_{\geq 0}$.
  b. Define $\leq$ and the operations on $\mathbb{Z}_{\geq 0}$.
  c. Show that $\mathbb{Z}_{\geq 0}$ is an ordered monoid.

(3)
  a. Define $\mathbb{Z}$.
  b. Define $\leq$ and the operations on $\mathbb{Z}$.
  c. Show that $\mathbb{Z}$ is an ordered ring.

(4) Define the clock **[???] IS THIS CORRECT?** monoid and show that it is a ring.

(5)
  a. Define $\mathbb{Q}$.
  b. Define $\leq$ on $\mathbb{Q}$ and the operations on $\mathbb{Q}$.
  c. Show that $\mathbb{Q}$ is an ordered field.

(6) Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be fields. Let $f : \mathbb{F}_1 \to \mathbb{F}_2$ be a function such that if $x, y \in \mathbb{F}_1$, then $f(xy) = f(x)f(y)$ and $f(x+y) = f(x) + f(y)$.

  a. Show that $f(0) = 0$.
  b. Show that $f(1) = 1$.
  c. Show that f is injective.

(7) Defive a function $f : \mathbb{Q} \to \mathbb{R}$ such that if $xy \in \mathbb{Q}$ then $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$.

a. Show that $f(1/8) = 0.125$.
b. Show that f is injective.
c. Show that f is not surjective.

(8)
a. Define $\mathbb{R}$.
b. Define $\leq$ on $\mathbb{R}$ and the operations on $\mathbb{R}$.
c. Show that $\mathbb{R}$ is an ordered field.

(9)
a. Define $\mathbb{Q}[x]$.
b. Define the operations on $\mathbb{Q}[x]$.
c. Show that $\mathbb{Q}[x]$ is an field.

(10)
a. Define $\mathbb{Q}(x)$.
b. Define the operations on $\mathbb{Q}(x)$.
c. Show that $\mathbb{Q}(x)$ is an field.

(11)
a. Define $\mathbb{Q}[[x]]$.
b. Define the operations on $\mathbb{Q}[[x]]$.
c. Show that $\mathbb{Q}[[x]]$ is an field.

(12)
a. Define $\mathbb{Q}((x))$.
b. Define the operations on $\mathbb{Q}((x))$.
c. Show that $\mathbb{Q}((x))$ is an field.

(13) State and prove the Pythagorean Theorem.

(14) Prove that there does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

(15)
a. Define $\left|\ \right|$ on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.
b. Define a metric space.
c. Show that $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are metric spaces.

(16)
a. Define $\mathbb{R}^7$.
b. Define $\left|\ \right|$ on $\mathbb{R}^7$.
c. Show that $\mathbb{R}^7$ is a metric space.

(17) Let X be a metric space. Define the metric space topology on X.

(18)
a. Define inverse function.
b. Define bijective.

c. Let $f : S \to T$ be a function. Prove that the inverse finction to f exists if and only if f is bijective.

(19) Write $\frac{1}{1-x}$ as an element of $\mathbb{Q}[[x]]$.

(20)
a. Define e^x.
b. Show that $e^0 = 1$
c. Show that $e^x e^y = e^{x+y}$.
d. Show that $e^{-x} = \frac{1}{e^x}$.

(21)
a. Define $\log x$.
b. Show that $\log(xy) = \log x + \log y$.
c. Show that $\log(1) = 0$.
d. Show that $\log(1/x) = -\log x$.

(22) Write $\frac{1}{1+x}$ as an element of $\mathbb{Q}[[x]]$.

(23) Write $\log(1+x)$ as an element of $\mathbb{Q}[[x]]$.

(24) Write $\frac{1}{1+x^2}$ as an element of $\mathbb{Q}[[x]]$.

(25) Write $\arctan x$ **[???] INSTEAD OF TAN^{-1}** as an element of $\mathbb{Q}[[x]]$.

(26) Prove that there is a unique function $D_x : \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ such that if $a, b \in \mathbb{Q}$ and $a, b \in \mathbb{Q}[[x]]$ then

a. $D_x(af + bg) = aD_x(f) + bD_x(g)$,
b. $D_x(fg) = fD_x(g) + D_x(f)g$, and
c. $D_x(x) = 1$.

(27) Let $p \in \mathbb{Q}[[x]]$. Prove that there is a unique function $D_p : \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ such that if $a, b \in \mathbb{Q}$ and $a, b \in \mathbb{Q}[[x]]$ then

a. $D_p(af + bg) = aD_p(f) + bD_p(g)$, **[???] I ASSUME THIS IS WHAT IS MEANT.**
b. $D_p(fg) = fD_p(g) + D_p(f)g$, and **[???] I ASSUME THIS IS WHAT IS MEANT.**
c. $D_p(x) = p$.

(28) Assume that $f = a_0 + a_1 x + a_2 x^2 + \ldots \in \mathbb{Q}[[x]]$. Show that $a_n = \frac{1}{n!} \left(D_x^n f\right)\big|_{x=0}$.

(29) Let D_x be as in problem (26) above. Show that if $n \in \mathbb{Z}_{>0}$ then $D_x(x^n) = nx^{n-1}$.

(30) Show that if $n \in \mathbb{Z}_{>0}$ then $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.

(31) Assume $D_x f = f$ and $f = 1 + a_1 + a_2 x^2 + \ldots \in \mathbb{Q}[[x]]$. Compute the a_n.

(32) Assume f and g are in $\mathbb{Q}[[x]]$ and that $D_x f = g$, $D_x g = -f$, $f(0) = 1$, and $g(0) = 1$. Compute f and g.

(33) Write $(1+x)^{1/2}$ as an element of $\mathbb{Q}[[x]]$.

(34) Write $(1+x)^7$ as an element of $\mathbb{Q}[[x]]$.

(35) Define Pascal's triangle and explain its relation to $x+y$, $(x+y)^2$, $(x+y)^3$, $\ldots$.

(36) Let S be a set. Define the power set of S. Show that $\supseteq$ is a partial order on the power set of S.

(37) For $x, y \in \mathbb{Z}_{\geq 0}$ define $x|y$ if there exists $n \in \mathbb{Z}_{>0}$ such that $xn = y$ **[???] DIFFERS FROM SHEET**. Show that $|$ is a partial order on $\mathbb{Z}_{>0}$.

(38) Give an example of a partially ordered set S and a subset $E \subseteq S$ such that E has a maximum which is not an upper bound.

(39)
   a. Define $\sup(E)$.
   b. Give an example of when $\sup(E)$ does not exist.
   c. Show that if $\sup(E)$ exists then it is unique.

(40)
   a. Define $\inf(E)$.
   b. Give an example of when $\inf(E)$ does not exist.
   c. Show that if $\inf(E)$ exists then it is unique.

(41) Show that $\mathbb{Z}_{>0}$ as a subset of $\mathbb{R}$ is not bounded above.

(42) As a subset of $\mathbb{Q}$ find $\sup\{x \in \mathbb{Q} \mid x^2 < 2\}$.

(43) Show that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

(44) Show that $\mathrm{Card}(\mathbb{Z}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

(45) Show that $\mathrm{Card}\big((0, 1]_{\mathbb{Q}}\big) = \mathrm{Card}(\mathbb{Z}_{>0})$.

(46) Show that $\mathrm{Card}\big((0, 1]_{\mathbb{R}}\big) = \mathrm{Card}(\mathbb{Z}_{>0})$.

(47) Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ and $\mathrm{Card}(T) = \mathrm{Card}(U)$ then $\mathrm{Card}(S) = \mathrm{Card}(U)$.

(48) Show that if $\mathrm{Card}(S) = \mathrm{Card}(T)$ then $\mathrm{Card}(T) = \mathrm{Card}(S)$.

(49) Satte and prove Lagrange's identity.

(50)
a. Define $\left| \right|$ and $\langle \ , \ \rangle$ on $\mathbb{R}^n$.
b. Prove that if $x, y \in \mathbb{R}$ **[???] OR R^n ?** then $\langle x, y\rangle \le |x||y|$.

(51)
a. Define $\left| \right|$ on $\mathbb{R}^n$.
b. Prove that if $x, y \in \mathbb{R}$ **[???] OR R^n ?** then $|x + y| \le |x||y|$.

(52)
a. Define ordered field.
b. Let $\mathbb{F}$ be an ordered field. Let $x, y \in \mathbb{F}$ with $x \ge 0$ and $y \ge 0$. Show that $x \le y$ if and only if $x^2 \le y^2$.

(53) Find $\lim_{n\to\infty} \frac{1}{n}$.

(54) Find $\lim_{n\to\infty} (-1)^{n-1}$.

(55) Find $\lim_{n\to\infty} n$.

(56) Let $x \in \mathbb{R}$. Find $\lim_{n\to\infty} x^n$.

(57) Let $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$. Find $\sup a_n$, $\inf a_n$, $\limsup a_n$ and $\liminf a_n$.

(58) Show that if (a_n) converges then (a_n) is Cauchy.

(59) Find $\sum_{n=0}^{\infty} (-1)^n$.

(60) Find $\sum_{n=0}^{\infty} x^n$.

(61) Find $\sum_{n=0}^{\infty} 1^n$.

(62) Find $\sum_{n=1}^{\infty} \frac{1}{n}$.

(63) Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(64) Show that if $k > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{k}}}$ converges.

(65) Show that if $k < 1$ **[???] OR EQUAL?** then $\sum_{n=1}^{\infty} \frac{1}{n^{k}}$ diverges.

(66) Find $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$.

(67) Find $\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{n}$.

(68) Find $\lim_{x \to 0} \frac{\log(1 + x)}{x}$.

(69) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{-1^{n-1}}{n} x^{n}$.

(70) Prove using the definition of the limit, that $\lim_{n \to \infty} \frac{n^2 - 1}{2n^2 + 3} = \frac{1}{2}$.

(71) If you borrow \$500 on your credit card at 14% interest find the amounts due at the end of two years if the interest is compounded

a. annually,
b. quarterly,
c. monthly,
d. daily,
e. hourly,
f. every second,
g. every nanosecond, and
h. continuously.

(72) Find a **[???] THE?** Taylor series for $\log(1 + x)$.

(73) Find $\lim_{n \to \infty} \frac{\log\left(1 + \frac{0.14}{n}\right)}{\frac{0.14}{n}}$.

(74) Find $\lim_{n \to \infty} 500\left(1 + \frac{0.14}{n}\right)^{2n}$.

(75) Explain Picard iteration.

(76) Explain Newton iteration.

(77) Define contractive sequence.

(78) Let (a_n) be a contractive sequence. Show that

$$|a_{n+1} - a_n| \le \alpha^{n+1} |a_2 - a_1|$$

where α is the contractive constant.

(79) Define topology and topological space.

(80) In $\mathbb{R}$, for each of the following intervals, determine whether it is open and whether it is closed:

a. (a, b)
b. $[a, b)$
c. $(a, b]$
d. $[a, b]$
e. $(-\infty, b)$
f. (a, ∞)

(81) Define open set and closed set.

(82) Define interior, closure, interior point and close point.

(83) Define neighbourhood of x.

(84) Let X be a topological space and let $E \subseteq X$.

a. Show that the interior of E is the set of interior points of E.
b. Show that the closure of E is the set of close points of E.

(85) Define continuous function between topological spaces.

(86) Define differentiable at $x = c$ and derivative at $x = c$.

(87) Define connected.

(88) Let X and Y be topological spaces. Assume $f : X \to Y$ is continuous. Show that if X is connected than $f(X)$ is connected.

(89) Define ε-ball.

(90) Define the **[???] QUALIFY?** topology on a metric space.

(91) Define the topology on $\mathbb{R}$ and $\mathbb{R}^n$.

(92) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$. Let $c \in [a, b]$ and assume $f'(c)$ exists and $g'(c)$ exists. Show that

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

(93) Carefully state and prove the intermediate value theorem.

(94) Carefully state and prove the mean value theorem.

(95) Define compact.

(96) Show that if $f : X \to Y$ is a continuous function and X is compact then $f(X)$ is compact.

(97) Let X be a metric space and $E \subseteq X$. Show that if E is compact then E is closed and bounded.

(98) Let $X = \mathbb{R}^n$ and $E \subseteq X$. Show that E is compact if and only if E is closed and bounded.

(99) Define bounded (for a subset of a metric space).

(100) Assume $f : [a, b] \to \mathbb{R}$ is continuous. Show that there exists $c \in [a, b]$ such that if $x \in [a, b]$ then $f(x) \leq f(c)$.

(101) Give an example of a continuous and differentiable function $f : [a, b] \to \mathbb{C}$ such that $f(a) = f(b)$ but $f'(x)$ never equals zero.

(102) Carefully state and prove l'Hôpital's rule.

(103) Evaluate $\lim_{x \to 0} \frac{5x}{x}$.

(104) Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

(105) Explain why l'Hôpital's rule works.

(106) Define the Riemann integral, the trapezoidal integral and Simpson's integral.

(107) Evaluate $\int_0^2 e^x dx$ using the definition of the Riemann integral.

(108) Evaluate $\int_{-1}^{1} \frac{1}{x^2} dx$ using the definition of the Riemann integral.

(109) Discuss $\int_{-1}^{1} \frac{1}{x^2} dx$ from the point of view of the Fundamental Theorem of Calculus.

(110) State the Fundamental Theorem of Calculus and explain why it is true.

(111) Define the improper integrals and give examples.

(112) Calculate $\int_0^\infty \frac{dx}{1 + x^2}$.

(113) Let $p \in \mathbb{R}, p > 1$. Compute $\int_1^\infty \frac{dx}{x^p}$.

(114) Evaluate $\int_1^\infty \frac{dx}{x}$.

(115) Let $p \in \mathbb{R}, 0 < p < 1$. Compute $\int_1^\infty \frac{dx}{x^p}$.

(116) Evaluate $\int_0^1 \frac{1}{x^{1/2}}$.

(117) Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}$.

(118) Define *converges pointwise* and *converges uniformly* and give examples.

(119) Graph the following functions.

a. $y = 1$
b. $y = 1 + \mathrm{x}$
c. $y = 1 + \mathrm{x} + \frac{x^2}{2}$
d. $y = 1 + \mathrm{x} + \frac{x^2}{2} + \frac{x^3}{6}$
e. $y = e^x$

(120) Give an example of a sequence of functions $f : [a, b] \to \mathbb{R}$ that converges pointwise but not uniformly.

(121) Show that the sequence of functions $f : [0, 1] \to \mathbb{R}$ given by $f_n(x) = \frac{1}{nx+1}$ converges pointwise, but not uniformly.

(122) What is the error in a trapezoidal approximation to $\int_a^b f(x)dx$?

(123) What is the error in a Simpson approximation to $\int_a^b f(x)dx$?

(124) Find ln(2) to within 0.01 using a trapezoidal approximation.

(125) Find ln(2) to within 0.01 using a Taylor series.

(126) Approximate $\sqrt{17}$ to within 0.0001 using Taylor series.

(127) State the Stone-Weierstrass theorem.

(128) Define trigonometric series.

(129) Compute $\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}dx$.

(130) Let $k, l \in \mathbb{Z}$. Compute $\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}e^{-ilx}dx$.

(131) Assume $f(x) = c_0 + c_1 e^{ix} + c_{\text{-}1} e^{-ix} + c_2 e^{2ix} + c_{\text{-}2} e^{-2ix} + \cdots$. Show that $c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$.

(132) Find the expansion of x^2 as a trigonometric series.

(133) Show that $\frac{\pi^2}{12} = \sum_{k=1}^{\infty} (\text{-}1)^{k-1} \frac{1}{k^2}$.

(134) Let $n \in \mathbb{Z}_{>0}$. Find $\lim_{x \to \infty} x^n e^{-x}$.

(135) Let $\alpha \in \mathbb{R}_{>0}$. Find $\lim_{x \to 0} x^{-\alpha} \ln x$.

(136) Let $p \in \mathbb{R}_{>0}$. Find $\lim_{n \to \infty} \frac{1}{n^p}$.

(137) Let $p \in \mathbb{R}_{>0}$. Find $\lim_{n \to \infty} p^{1/n}$.

(138) Find $\lim_{n \to \infty} n^{1/n}$.

(139) Let $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}$. Find $\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n}$.

(140) Assume $|x| < 1$. Find $\lim_{n \to \infty} x^n$.

(141) Find $\lim_{x \to 0} \frac{e^x - 1}{x}$.

(142) Find $\lim_{x \to 0} \frac{\sin x}{x}$.

(143) Find $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$.

(144) Find $\lim_{x \to 0} \frac{\log(1+x)}{x}$.

2. References [PLACEHOLDER]

[BG] A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmanian*, Duke Math. J. **107** no. 3, (2001), 561-575; arXiv:math/9909077v2, MR1828302 (2002e:20083)