Problem Set

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au
and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Last updates: 19 October 2009

Problem Sets

1. The Fundamental Theorem of Calculus

(1) What does

$$\int_a^b f(x)dx$$

mean?

(2) How does one usually calculate

$$\int_a^b f(x)dx \ ?$$

Give an example which shows that this method does not always work. Why doesn't it?

(3) Give an example which shows that

$$\int_a^b f(x)dx$$

is not always the true area under $f(x)$ between a and b even if $f(x)$ is contunuous between a and b.

(4) What is the *Fundamental Theorem of Calculus*?

(5) Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

(6) Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

(7) Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this, compute an area by summing up the areas of tiny boxes and then show that applying the Fundamental Theorem of Calculus gives the same result.

2. Where is a Function Continuous?

(1) For which values of x is the function $f(x) = x^2 + 3x + 4$ contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(2) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(3) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(4) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(5) Determine the value of k for which the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0, \end{cases}$$

contunuous at $x = 0$. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(6) For which values of x is the function

$$f(x) = \begin{cases} x - 1, & \text{if } 1 \leq x < 2, \\ 2x - 3, & \text{if } 2 \leq x \leq 3, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(7) For which values of x is the function

$$f(x) = \begin{cases} \cos x, & \text{if } x \geq 0, \\ -\cos x, & \text{if } x < 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(8) For which values of x is the function

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(9) Determine the value of a for which the function

$$f(x) = \begin{cases} ax + 5, & \text{if } x \leq 2, \\ x - 1, & \text{if } x > 2, \end{cases}$$

contunuous at $x = 2$. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(10) For which values of x is the function

$$f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \le x \le 1, \\ 2 - x, & \text{if } x > 1, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(11) For which values of x is the function $f(x) = 2x - \lfloor x \rfloor$ contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(12) Find the value of a for which the function

$$f(x) = \begin{cases} 2x - 1, & \text{if } x < 2, \\ a, & \text{if } x = 2, \\ x + 1, & \text{if } x > 2, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(13) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{\lfloor x - a \rfloor}{x - a}, & \text{if } x \ne a, \\ 1, & \text{if } x = a, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(14) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{x - \lfloor x \rfloor}{2}, & \text{if } x \ne 0, \\ 2, & \text{if } x = 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(15) For which values of x is the function

$$f(x) = \begin{cases} \sin x, & \text{if } x < 0, \\ \text{x}, & \text{if } x \ge 0, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(16) For which values of x is the function

$$f(x) = \begin{cases} \dfrac{x^n - 1}{x - 1}, & \text{if } x \ne 1, \\ \text{n}, & \text{if } x = 1, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(17) Explain how you know $f(x) = \cos x$ is continuous for all values of x. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(18) Explain how you know $f(x) = \cos|x|$ is continuous for all values of x. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(19) Explain how you know $f(x) = \lfloor x \rfloor$ is continuous for all values of x. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(20) For what values of x is the function

$$f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1, \\ 4, & \text{if } x = 1, \end{cases}$$

contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(21) For what values of x is the function $f(x) = |x| + |x - 1|$, $-1 \leq x \leq 2$, contunuous? Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.

3. Existence of Limits

(1) Explain why $\lim_{x \to 0}\left(\frac{1}{x}\right)$ does not exist.

(2) Explain why $\lim_{x \to \pi/2} \tan(x)$ does not exist.

(3) Explain why $\lim_{x \to \pi/2} \sec(x)$ does not exist.

(4) Explain why $\lim_{x \to 0} \csc(x)$ does not exist.

(5) Explain why $\lim_{x \to -1} \ln(x)$ does not exist.

(6) Explain why $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

(7) Explain why $\lim_{x \to \infty} \cos(x)$ does not exist.

(8) Explain why $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist,

where

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0. \\ -1, & \text{if } x < 0 \end{cases}$$

(9) Explain why $\lim_{x \to 0} 2^{1/x}$ does not exist.

(10) Explain why $\lim_{x \to 1} 2^{1/(1-x)}$ does not exist.

4. Increading, Decreasing and Concave Functions

(1) What does it mean for a function $f(x)$ to be continuous at $x = a$? Explain how to test if a function is continuous at $x = a$.

(2) What does it mean for a function $f(x)$ to be differentiable at $x = a$? Explain how to test if a function is differentiable at $x = a$.

(3) What does $(df/dx)\big|_{x=a}$ indicate about the graph of $y = f(x)$? Explain why this is true.

5. Evaluating Limits when $x \to 0$

(1) Evaluate $\lim_{x \to 0} \left(x^2 - 2\right)^2 + 6$.

(2) Evaluate $\lim_{x \to 0} \dfrac{5x}{x}$.

(3) Evaluate $\lim_{x \to 0} \dfrac{17x}{2x}$.

(4) Evaluate $\lim_{x \to 0} \dfrac{-317x}{422x}$.

(5) Evaluate $\lim_{x \to 0} \dfrac{-317x - 3}{422x + 5}$.

(6) Evaluate $\lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

(7) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1 + x + x^2} - 1}{x}$.

(8) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$.

(9) Evaluate $\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}\right)$.

(10) Evaluate $\lim\limits_{x\to 0}\dfrac{2x}{\sqrt{a+x}-\sqrt{a-x}}$.

(11) Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$.

(12) Evaluate $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$.

(13) Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x}+e^{-x}-2}{x^2}$.

(14) Evaluate $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$, when $f(x)=\sqrt{ax+b}$.

(15) Evaluate $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$, when $f(x)=(mx+c)^n$.

6. Evaluating Limits when $x \to a$

(1) Evaluate $\lim\limits_{x\to 1}\left(6x^2-4x+3\right)$.

(2) Evaluate $\lim\limits_{x\to 7}\dfrac{x^2-49}{x-7}$.

(3) Evaluate $\lim\limits_{x\to 2}\dfrac{x^2-6x+8}{x-2}$.

(4) Evaluate $\lim\limits_{x\to -5}\dfrac{2x^2+9x-5}{x+5}$.

(5) Evaluate $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$.

(6) Evaluate $\lim\limits_{x\to 3}\dfrac{x^2-4x+3}{x^2-2x-3}$.

(7) Evaluate $\lim\limits_{x\to -2}\dfrac{x^3+8}{x+2}$.

(8) Evaluate $\lim\limits_{x\to 3}\dfrac{x^4-81}{x-3}$.

(9) Evaluate $\lim_{x \to 5} \dfrac{x^5 - 3125}{x - 5}$.

(10) Evaluate $\lim_{x \to a} \dfrac{x^{12} - a^{12}}{x - a}$.

(11) Evaluate $\lim_{x \to a} \dfrac{x^{5/2} - a^{5/2}}{x - a}$.

(12) Evaluate $\lim_{x \to a} \dfrac{(x + 2)^{5/3} - (a + 2)^{5/3}}{x - a}$.

(13) Evaluate $\lim_{x \to 4} \dfrac{x^3 - 64}{x^2 - 16}$.

(14) Evaluate $\lim_{x \to 2} \dfrac{x^5 - 32}{x^3 - 8}$.

(15) Evaluate $\lim_{x \to 1} \dfrac{x^n - 1}{x - 1}$.

(16) Evaluate $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a}$.

(17) Evaluate $\lim_{x \to 2} \dfrac{\sqrt{3 - x} - 1}{2 - x}$.

(18) Evaluate $\lim_{x \to a} \dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}}$.

(19) Evaluate $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$.

7. Evaluating Limits when $x \to \infty$

(1) Evaluate $\lim_{x \to \infty} \dfrac{x + 2}{x - 2}$.

(2) Evaluate $\lim_{x \to \infty} \dfrac{3x^2 + 2x - 5}{5x^2 + 3x + 1}$.

(3) Evaluate $\lim_{x \to \infty} \dfrac{x^2 - 7x + 11}{3x^2 + 10}$.

(4) Evaluate $\lim_{x \to \infty} \dfrac{2x^3 - 5x + 7}{7x^3 + x^2 - 6}$.

(5) Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^3 - 5x + 7}{7x^3 + x^2 - 6}$.

(6) Evaluate $\lim\limits_{x\to\infty} \dfrac{(3x-1)(4x-5)}{(x-6)(x-3)}$.

(7) Evaluate $\lim\limits_{n\to\infty} \left(+\dfrac{1}{3^2} + \dfrac{1}{3^3} + \ldots + \dfrac{1}{3^n}\right)$.

(8) Evaluate $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{4x^2-1} - 1}$.

(9) Evaluate $\lim\limits_{x\to-\infty} 2^x$.

(10) Evaluate $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$.

(11) Evaluate $\lim\limits_{t\to\infty} \dfrac{t+1}{t^2+1}$.

(12) Evaluate $\lim\limits_{n\to\infty} \sqrt{n^2+1} + n$.

(13) Evaluate $\lim\limits_{n\to\infty} \sqrt{n^2+n} + n$.

8. Limits with Exponential and Logarithm Functions

(1) Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$.

(2) Evaluate $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x}$.

(3) Evaluate $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$.

(4) Evaluate $\lim\limits_{x\to 0} (1+x)^{1/x}$.

(5) Evaluate $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$.

(6) Evaluate $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$.

(7) Evaluate $\lim\limits_{x\to-\infty} 2^x$.

(8) Explain why $\lim\limits_{x\to -1} \ln(x)$ does not exist.

(9) Explain why $\lim_{x \to 0} 2^{1/x}$ does not exist.

(10) Explain why $\lim_{x \to 1} 2^{1/(x-1)}$ does not exist.

(11) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ where $f(x) = e^{\sqrt{x}}$.

(12) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ where $f(x) = \ln(ax+b)$.

(13) Evaluate $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ where $f(x) = x^x$.

9. Limits with Trigonometric Functions

(1) Evaluate $\lim_{x \to 0} \dfrac{\sin 3x}{4x}$.

(2) Evaluate $\lim_{x \to 0} \dfrac{\sin x \cos x}{3x}$.

(3) Evaluate $\lim_{x \to 0} \dfrac{\tan x}{x}$.

(4) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos x}{\sin^2 x}$.

(5) Evaluate $\lim_{x \to 0} \dfrac{\tan ax}{\tan bx}$.

(6) Evaluate $\lim_{x \to 0} \dfrac{\sin(x/4)}{x}$.

(7) Evaluate $\lim_{x \to 0} \dfrac{\sin mx}{\tan nx}$.

(8) Evaluate $\lim_{\theta \to 0} \dfrac{1 - \cos 6\theta}{\theta}$.

(9) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{3\tan^2 x}$.

(10) Evaluate $\lim_{x \to 0} \dfrac{\cos^2 x}{1 - \sin x}$.

(11) Evaluate $\lim_{x \to 0} \dfrac{\tan 2x - x}{3x - \sin x}$.

(12) Evaluate $\lim_{x \to a} \dfrac{\sin x - \sin a}{x - a}$.

(13) Evaluate $\lim_{x\to 0}\dfrac{\sin 5x - \sin 3x}{\sin x}$.

(14) Evaluate $\lim_{x\to 0}\dfrac{\tan 3x - 2x}{3x - \sin^2 x}$.

(15) Evaluate $\lim_{x\to 0}\dfrac{x^2 - \tan 2x}{\tan x}$.

(16) Evaluate $\lim_{x\to \pi/4}\dfrac{1 - \tan x}{x - \pi/4}$.

(17) Evaluate $\lim_{x\to 0}\dfrac{\tan(x/2)}{3x}$.

(18) Evaluate $\lim_{x\to 0}\dfrac{1 - \cos 2x + \tan^2 x}{x \sin x}$.

(19) Show that if

$\lim_{x\to 0} kx \csc x = \lim_{x\to 0} x \csc kx$,

then

$k = \pm 1$.

(20) Evaluate $\lim_{h\to 0}\dfrac{\sin(a+h) - \sin a}{h}$.

(21) Evaluate $\lim_{h\to \infty}\dfrac{\cos(\pi/h)}{h-2}$.

10. Limits with Inverse Trigonometric Functions

(1) Evaluate $\lim_{x\to 1}\dfrac{1-x}{\arccos^2 x}$.

(2) Evaluate $\lim_{x\to 1/\sqrt{2}}\dfrac{x - \cos(\arcsin x)}{1 - \tan(\arcsin x)}$.

(3) Evaluate $\lim_{x\to 0}\dfrac{x\left(1-\sqrt{1-x^2}\right)}{\arcsin^3(x)\sqrt{1-x^2}}$.

(4) Evaluate $\lim_{x\to 1}\dfrac{1-x}{\pi - 2\arcsin x}$.

(5) Evaluate $\lim_{x\to 1}\dfrac{\arctan 2x}{\sin 3x}$.

11. L'Hôpital's Rule

(1) State L'Hôpital's rule and give an example which shows how it is used.

(2) Explain why L'Hôpital's rule works. Hint: Expand the numerator and the denominator in terms of Δx.

(3) Give three examples which illustrate that a limit problem that looks like it is coming out to $0/0$ could really be getting closer and closer to almost anything and must be looked at in a different way.

(4) Give three examples which illustrate that a limit problem that looks like it is coming out to 1^{∞} could really be getting closer and closer to almost anything and must be looked at in a different way.

(5) Give three examples which illustrate that a limit problem that looks like it is coming out to 0^0 could really be getting closer and closer to almost anything and must be looked at in a different way.

(6) Evaluate $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$.

(7) Evaluate $\lim\limits_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$.

(8) Evaluate $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$.

(9) Evaluate $\lim\limits_{x \to \pi} \dfrac{\tan x}{x - \pi}$.

(10) Evaluate $\lim\limits_{x \to 3\pi/2} \dfrac{\cos x}{x - (3\pi/2)}$.

(11) Evaluate $\lim\limits_{x \to 0^+} \dfrac{\ln x}{\sqrt{x}}$.

(12) Evaluate $\lim\limits_{x \to \infty} \dfrac{(\ln x)^3}{x^2}$.

(13) Evaluate $\lim\limits_{x \to 0} \dfrac{6^x - 2^x}{x}$.

(14) Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x - \left(x^2/2\right)}{x^3}$.

(15) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^3}$.

(16) Evaluate $\lim\limits_{x \to \infty} \dfrac{\ln(1 + e^x)}{5x}$.

(17) Evaluate $\lim_{x\to 0} \frac{\tan \alpha x}{x}$.

(18) Evaluate $\lim_{x\to 0} \frac{2x - \arcsin x}{2x - \arccos x}$.

(19) Evaluate $\lim_{x\to 0^+} \sqrt{x} \ln x$.

(20) Evaluate $\lim_{x\to \infty} e^{-x} \ln x$.

(21) Evaluate $\lim_{x\to \infty} x^3 e^{-x^2}$.

(22) Evaluate $\lim_{x\to \infty} (x - \pi)\cot x$.

(23) Evaluate $\lim_{x\to 0} x^{-4} - x^{-2}$.

(24) Evaluate $\lim_{x\to 0} x^{-1} - \csc x$.

(25) Evaluate $\lim_{x\to \infty} x - \sqrt{x^2 - 1}$.

(26) Evaluate $\lim_{x\to \infty} \left(\frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1}\right)$.

(27) Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

(28) Evaluate $\lim_{x\to 0} (1 - 2x)^{1/x}$.

(29) Evaluate $\lim_{x\to \infty} \left(1 + 3/x + 5/x^2\right)^x$.

(30) Evaluate $\lim_{x\to \infty} x^{1/x}$.

(31) Evaluate $\lim_{x\to 0^+} (\cot x)^{\sin x}$.

(32) Evaluate $\lim_{x\to \infty} \left(\frac{x}{x - 1}\right)^x$.

(33) Evaluate $\lim_{x\to 0^+} (-\ln x)^x$.

12. Sets

(1) DeMorgan's Laws. Let A, B and C be sets. Show that

a. $(A \cup B) \cup C = A \cup (B \cup C)$,

b. $A \cup B = B \cup A$,

c. $A \cup \emptyset = A$,

d. $(A \cap B) \cap C = A \cap (B \cap C)$,

e. $A \cap B = B \cap A$, and

f. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

13. Functions

(1) Let S, T and U be sets and let $f : S \to T$ and $g : T \to U$ be functions. Show that

a. if f and g are injective then $g \circ f$ is injective,

b. if f and g are surjective then $g \circ f$ is surjective, and

c. if f and g are bijective then $g \circ f$ is bijective.

(2) Let $f : S \to T$ be a function and let $U \subseteq S$. The **image** of U under f is the subset of T given by

$$f(U) = \{f(u) | u \in U\}.$$

Let $f : S \to T$ be a function. The **image** of U under f is the subset of T given by

$$\mathrm{im}\, U = \{f(s) | s \in S\}.$$

Note that $\mathrm{im}\, f = f(S)$.

Let $f : S \to T$ be a function and let $V \subseteq T$. The **inverse image** of V under f is the subset of S given by

$$f^{-1}(V) = \{s \in S \,|\, f(s) \in V\}.$$

Let $f : S \to T$ be a function and let $t \in T$. The **fiber** of f over t is the subset of S given by

$$f^{-1}(t) = \{s \in S \,|\, f(s) = t\}.$$

Let $f : S \to T$ be a function. Show that the set $F = \left\{f^{-1}(t) | t \in T\right\}$ of fibers of the map f is a partition of S.

(3) a. Let $f : S \to T$ be a function. Define

$$\begin{array}{rcl} f' : S & \longrightarrow & \mathrm{im} f \\ s & \longmapsto & f(s). \end{array}$$

Show that the map f' is well defined and surjective.

b. Let $f : S \to T$ be a function and let $F = \left\{f^{-1}(t) | t \in \mathrm{im}\, f\right\} = \left\{f^{-1}(t) | t \in T\right\} \setminus \emptyset$ be the set of nonempty fibers of the map f. Define

$$\begin{array}{rccc} \hat{f}: & F & \longrightarrow & T \\ & f^{-1}(t) & \longmapsto & t \end{array}.$$

Show that the map $\hat{f}$ is well defined and injective.

c. Let $f: S \to T$ be a function and let $F = \left\{ f^{-1}(t) \mid t \in \operatorname{im} f \right\} = \left\{ f^{-1}(t) \mid t \in T \right\} \setminus \varnothing$ be the set of nonempty fibers of the map f. Define

$$\begin{array}{rccc} \hat{f}': & F & \longrightarrow & \operatorname{im} T \\ & f^{-1}(t) & \longmapsto & t \end{array}.$$

Show that the map $\hat{f}'$ is well defined and bijective.

(4) Let be a set. The **power set** of S, 2^S, is the set of all subsets of S.

Let S be a set and let $\{0, 1\}^S$ be the set of all functions $f: S \to \{0, 1\}$. Given a subset $T \subseteq S$ define a function $f_T: S \to \{0, 1\}$ by

$$f_T(s) = \begin{cases} 0, & \text{if } s \notin T, \\ 1, & \text{if } s \in T. \end{cases}$$

Show that the map

$$\begin{array}{rccc} \phi: & 2^S & \longrightarrow & \{0, 1\}^S \\ & T & \longmapsto & f_T \end{array}$$

is a bijection.

(5) Let $\circ: S \times S \to S$ be an associative operation on a set S. An **identity** for $\circ$ is an element $e \in S$ such that $e \circ s = s \circ e = s$ for all $s \in S$.

Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. A **left inverse** for s is an element $t \in S$ such that $t \circ s = e$. A **right inverse** for s is an element $t' \in S$ such that $s \circ t' = e$. An **inverse** for s is an element $s^{-1} \in S$ such that $s^{-1} \circ s = s \circ s^{-1} = e$.

a. Let $\circ$ be an operation on a set S. Show that if S contains an identity for $\circ$ then it is unique.

b. Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. Show that if s has an inverse then it is then it is unique.

(6) a. Let S and T be sets and let ι_S and ι_T be the identity maps on S and T respectively. Show that for any function $f: S \to T$,

$$\iota_T \circ f = f, \qquad \text{and}$$

$$f \circ \iota_S = f.$$

b. Let $f : S \to T$ be a function. Show that if an inverse function to f exists then it is unique. (Hint: The proof is very similar to the proof in Ex. 5b above.)

14. Ordered sets

(1) Show that if a greatest lower bound exists, then it is unique.

(2) Show that if S is a lattice then the intersection of two intervals is an interval.

(3) A poset S is **left filtered** if every subset E of S has an upper bound.

A poset S is **right filtered** if every subset E of S has an lower bound.

Let S be a poset and let E be a subset of S. A **minimal element** of E is an element $x \in E$ such that if $y \in E$ then $x \leq y$.

A poset S is **well ordered** if every subset E of S has a minimal element.

Show that every well ordered set is totally ordered.

(4) Show that there exist totally ordered sets that are not well ordered.

15. Graphs of the Basic Functions

(1) Graph $f(x) = |x|$.

(2) Graph $f(x) = \lfloor x \rfloor$.

(3) Graph $f(x) = 2$.

(4) Graph $f(x) = x$.

(5) Graph $f(x) = x^2$.

(6) Graph $f(x) = x^3$.

(7) Graph $f(x) = x^4$.

(8) Graph $f(x) = x^5$.

(9) Graph $f(x) = x^6$.

(10) Graph $f(x) = x^{100}$.

(11) Graph $f(x) = x^{-1}$.

(12) Graph $f(x) = x^{-2}$.

(13) Graph $f(x) = x^{-3}$.

(14) Graph $f(x) = x^{-4}$.

(15) Graph $f(x) = x^{-100}$.

(16) Graph $f(x) = e^x$.

(17) Graph $f(x) = \sin x$.

(18) Graph $f(x) = \cos x$.

(19) Graph $f(x) = \tan x$.

(20) Graph $f(x) = \cot x$.

(21) Graph $f(x) = \sec x$.

(22) Graph $f(x) = \csc x$.

(23) Graph $f(x) = \sqrt{x}$.

(24) Graph $f(x) = x^{1/3}$.

(25) Graph $f(x) = x^{1/4}$.

(26) Graph $f(x) = x^{1/5}$.

(27) Graph $f(x) = x^{1/6}$.

(28) Graph $f(x) = \dfrac{1}{\sqrt{x}}$.

(29) Graph $f(x) = x^{-1/3}$.

(30) Graph $f(x) = x^{-1/4}$.

(31) Graph $f(x) = \ln x$.

(32) Graph $f(x) = \arcsin x$.

(33) Graph $f(x) = \arccos x$.

(34) Graph $f(x) = \arctan x$.

(35) Graph $f(x) = \operatorname{arccot} x$.

(36) Graph $f(x) = \operatorname{arcsec} x$.

(37) Graph $f(x) = \operatorname{arccsc} x$.

16. Graphing Polynomials

(1) Let $f(x) = a$, where A is a constant.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(2) Let $f(x) = ax + b$ where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(3) Let $f(x) = a(x - c) + b$, where a, b and c is a constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.

e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.

g. Determine what the critical pionts of $f(x)$ are.

h. Determine what the points of inflection of $f(x)$ are.

i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(4)

$$\text{Let } f(x) = \begin{cases} 2 - x, & \text{if } x \geq 1, \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$$

a. Graph $f(x)$.

b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.

d. Determine where $f(x)$ is differentiable.

e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.

g. Determine what the critical pionts of $f(x)$ are.

h. Determine what the points of inflection of $f(x)$ are.

i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(5)

$$\text{Let } f(x) = \begin{cases} 2 + x, & \text{if } x > 0, \\ 2 - x, & \text{if } x \leq 0. \end{cases}$$

a. Graph $f(x)$.

b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.

d. Determine where $f(x)$ is differentiable.

e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.

g. Determine what the critical pionts of $f(x)$ are.

h. Determine what the points of inflection of $f(x)$ are.

i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(6)

$$\text{Let } f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ x^2 - 1, & \text{if } x \geq 1. \end{cases}$$

a. Graph $f(x)$.

b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.

d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(7) Let $f(x) = 2x - x^2$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(8) Let $f(x) = x - x^2 - 27$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(9) Let $f(x) = 3x^2 - 2x - 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.

g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(10) Let $f(x) = x^3$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(11) Let $f(x) = x^3 - x + 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(12) Let $f(x) = x^3 - x - 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(13) Let $f(x) = (x-2)^2(x-1)$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(14) Let $f(x) = 2x^3 - 21x^2 + 36x - 20$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(15) Let $f(x) = 2x^3 + x^2 + 20x$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(16) Let $f(x) = 1 - x^4$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(17) Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(18) Let $f(x) = \; = 3x^4 - 16x^3 + 18x^2$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(19) Let $f(x) = x^5 - 4x^4 + 4x^3$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(20) Let $f(x) = x^3(x-2)^2$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(21) Let $f(x) = (x-2)^4(x+1)^3(x-1)$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

17. Graphing Rational Functions

(1) Let $f(x) = y$ where $x^2 + y^2 = 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.

g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(2) Let $f(x) = \sqrt{1-x^2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(3) Let $f(x) = \sqrt{a^2-x^2}$, where a is a constant.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(4) Let $f(x) = y$, where $(x-h)^2 + (y-k)^2 = r^2$, where h, k and r are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(5) Let $f(x) = y$, where $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$, where h, k and r are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(6) Let $f(x) = y$ where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(7) Let $f(x) = y$ where $x = a\cos\theta$ and $y = b\sin\theta$, where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(8) Let $f(x) = (b/a)\sqrt{a^2 - x^2}$ where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(9) Let $f(x) = y$, where $x^2 - y^2 = 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(10) Let $f(x) = y$ where $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(11) Let $f(x) = ax^2 - b$, where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(12) Let $f(x) = y$, where $x = 2y^2 - 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(13) Let $f(x) = y$, where $x = \cos 2\theta$ and $y = \cos\theta$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(14) Let $f(x) = b\sqrt{x-a}$ where a and b are constants.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(15) Let $f(x) = \sqrt{x+2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(16) Let $f(x) = -\sqrt{x+2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(17) Let $f(x) = y$ where $y^2\left(x^2 - \mathrm{x}\right) = x^2 - 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.

i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(18) Let $f(x) = y$ where $x = \dfrac{y^2 - 1}{y^2 + 1}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(19) Let $f(x) = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(20) Let $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(21) Let $f(x) = x\sqrt{32 - x^2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(22) Let $f(x) = x\sqrt{1 - x^2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

18. Graphing Other Functions

(1) Let $f(x) = \lfloor x \rfloor$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(2) Let $f(x) = |x|$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(3) Let $f(x) = |x - 5|$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(4) Let $f(x) = \left|x^2 - 1\right|$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(5)

$$\text{Let } f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ \text{-}1, & \text{if } x < 0. \end{cases}$$

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(6) Let $f(x) = (x-1)^{1/3}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(7) Let $f(x) = x^{2/3}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(8) Let $f(x) = \dfrac{1}{(x-1)^{2/3}}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.

c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(9) Let $f(x) = x(1-x)^{2/5}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(10) Let $f(x) = x^{2/3}(6-x)^{1/3}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(11) Let $f(x) = y$, where $\sqrt{x} + \sqrt{y} = 1$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.

f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(12) Let $f(x) = y$, where $x^{2/3} + y^{2/3} = a^{2/3}$, where a is a constant.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(13) Let $f(x) = y$, where $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(14) Let $f(x) = \sin x$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.

i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(15) Let $f(x) = \sin 2x - x$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(16) Let $f(x) = \sin x - \cos x$, for $-\pi/3 < x < 0$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(17) Let $f(x) = 2\cos x - \sin 2x$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(18) Let $f(x) = \dfrac{\sin x}{x}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(19) Let $f(x) = \sin(1/x)$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(20) Let $f(x) = e^{-x}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(21) Let $f(x) = e^{1/x}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.

d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(22) Let $f(x) = e^{-x^2}$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(23) Let $f(x) = \ln\left(4 - x^2\right)$.

a. Graph $f(x)$.
b. Determine where $f(x)$ is defined.
c. Determine where $f(x)$ is continuous.
d. Determine where $f(x)$ is differentiable.
e. Determine where $f(x)$ is increasing and where it is decreasing.
f. Determine where $f(x)$ is concave up and where it is concave down.
g. Determine what the critical pionts of $f(x)$ are.
h. Determine what the points of inflection of $f(x)$ are.
i. Determine what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

19. Rolle's Theorem and the Mean Value Theorem

(1) State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

(2) State the mean value theorem and draw a picture which illustrates the statement of the theorem.

(3) Explain why Rolle's theorem is a *special case* of the mean value theorem.

(4) Verify Rolle's theorem for the function $f(x) = (x - 1)(x - 2)(x - 3)$ on the interval [1, 3].

(5) Verify Rolle's theorem for the function $f(x) = (x - 2)^2(x - 3)^6$ on the interval [2, 3].

(6) Verify Rolle's theorem for the function $f(x) = \sin x - 1$ on the interval $[\pi/2, 5\pi/2]$.

(7) Verify Rolle's theorem for the function $f(x) = e^{-x} \sin x$ on the interval $[0, \pi]$.

(8) Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.

(9) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in the interval [-1, 1] such that $\left.\dfrac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(10) Let $f(x) = (x - 1)^{-2}$. Show that $f(0) = f(2)$ but there is no number c in the interval [0, 2] such that $\left.\dfrac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(11) Discuss the applicability of Rolle's theorem when $f(x) = (x - 1)(2x - 3)$ on the interval $1 \leq x \leq 3$.

(12) Discuss the applicability of Rolle's theorem when $f(x) = 2 + (x - 1)^{2/3}$ on the interval $0 \leq x \leq 2$.

(13) Discuss the applicability of Rolle's theorem when $f(x) = \lfloor x \rfloor$ on the interval $-1 \leq x \leq 1$.

(14) At what point on the curve $y = 6 - (x - 3)^2$ on the interval [0, 6] is the tangent to the curve parallel to the x-axis?

(15) Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real solution.

(16) Show that a polynomial of degree three has at most three real roots.

(17) Verify the mean value theorem for the function $f(x) = x^{2/3}$ on the interval [0, 1].

(18) Verify the mean value theorem for the function $f(x) = \ln x$ on the interval $[1, e]$.

(19) Verify the mean value theorem for the function $f(x) = x$ on the interval [a, b], where a and b are constants.

(20) Verify the mean value theorem for the function $f(x) = lx^2 + mx + n$ on the interval [a, b], where l, m, n, a and b are constants.

(21) Show that the mean value theorem is not applicable to the function $f(x) = |x|$ in the interval [-1, 1].

(22) Show that the mean value theorem is not applicable to the function $f(x) = 1/x$ in the interval [-1, 1].

(23) Find the points on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining (1, -2) and (2, 2).

(24) If $f(x) = x(1 - \ln x)$, $x > 0$, show that $(a - b)\ln c = b(1 - \ln b) - a(1 - \ln a)$, where $0 < a < b$. **[???] FOR SOME c IN [a,b]?**

20. References [PLACEHOLDER]

[BG] A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmanian*, Duke Math. J. **107** no. 3, (2001), 561-575; arXiv:math/9909077v2, MR1828302 (2002e:20083)