620-295 Real Analysis with applications

Problem Sheet 6

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1. Trapezoidal and Simpson approximations

1. Determine the area of a trapezoid with left edge at $x = l$, right edge at $x = l + \Delta x$, left height $f(l)$, and right height $f(l + \Delta x)$.
2. Determine the area of a parabola topped slice with left edge at $x = l$, right edge at $x = l + 2\Delta x$, middle at $x = l + \Delta x$, left height $f(l)$, middle height $f(l + \Delta x)$, and right height $f(l + 2\Delta x)$.
3. Let N be a positive integer. Show that adding up N trapezoidal slices gives the approximation to $\int_a^b f(x)dx$ given by $\frac{\Delta x}{2}(f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 2f(b - \Delta x) + f(b))$, where $\Delta x = \frac{b-a}{N}$.
4. Let N be an even positive integer. Show that adding up N parabola topped slices gives the approximation to $\int_a^b f(x)dx$ given by $\frac{\Delta x}{2}(f(a) + 4f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 4f(b - \Delta x) + f(b))$, where $\Delta x = \frac{b-a}{N}$.
5. Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^2 (1 + x^2)dx$ and obtain a bound for the error.
6. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^2 (1 + x^2)dx$ and obtain a bound for the error.
7. Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.
8. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x}\,dx$ and obtain a bound for the error.
9. Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^{\pi/2} \sin x\,dx$ and obtain a bound for the error.

10. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^{\pi/2} \sin x \, dx$ and obtain a bound for the error.

11. Compute a trapezoidal approximation with $N = 4$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

12. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

13. Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^2 (1 + x^2) dx$ and obtain a bound for the error.

14. Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^2 (1 + x^2) dx$ and obtain a bound for the error.

15. Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^1 e^{-x} \, dx$ and obtain a bound for the error.

16. Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x} \, dx$ and obtain a bound for the error.

17. Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^{\pi/2} \sin x \, dx$ and obtain a bound for the error.

18. Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^{\pi/2} \sin x \, dx$ and obtain a bound for the error.

19. Compute a Simpson approximation with $N = 4$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

20. Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$ and obtain a bound for the error.

21. Let T_4 be the trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 \left(1 + x^2\right)^{-1} dx$. Show that $|f''(x)| \le 2$ for $x \in [0,\ 1]$ and that $|T_4 - \frac{1}{4}\pi| \le 1/96 < 0.0105$.

22. Use the trapezoidal approximation with $N = 4$ slices to approximate the integral $\log 2 = \int_1^2 x^{-1} \, dx$. Show that $0.6866 \le \log 2 \le 0.6958$.

23. Use Simpson's approximation with $N = 4$ slices to approximate $\log 2$. Show that $0.6927 \le \log 2 \le 0.6933$.

24. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x^2}\,dx$.

25. Compute a trapezoidal approximation with $N = 16$ slices for the integral $\int_0^1 e^{-x^2}\,dx$.

26. Compute a Simpson approximation with $N = 8$ slices for the integral $\int_0^1 e^{-x^2}\,dx$.

27. Compute a Simpson approximation with $N = 16$ slices for the integral $\int_0^1 e^{-x^2}\,dx$.

28. Compute a trapezoidal approximation with $N = 8$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx$.

29. Compute a trapezoidal approximation with $N = 16$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx$.

30. Compute a Simpson approximation with $N = 8$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx$.

31. Compute a Simpson approximation with $N = 16$ slices for the integral $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx$.

32. Derive the *midpoint approximation* for $\displaystyle\int_a^b f(x)\,dx$. With N slices it is obtained by adding up the areas of rectangles with height equal to the value of the function at the midpoint of the interval. Show that the error estimate is given by $\displaystyle\left|\int_a^b f(x)\,dx - M_N\right| \le \frac{(b-a)^3}{24n^2} M$, where M is an upper bound for $|f''(x)|$ on $[a, b]$.

2. Taylor approximations

1. Define the following and give an example of each:
   (a) converges pointwise
   (b) converges uniformly
2. Write a quadratic approximation for $f(x) = x^{1/3}$ near 8 and approximate $9^{1/3}$. Estimate the error and find the smallest interval that you can be sure contains the value.

3. Write a quadratic approximation for $f(x) = x^{-1}$ near 1 and approximate 1/1.02. Estimate the error and find the smallest interval that you can be sure contains the value.

4. Write a quadratic approximation for $f(x) = e^x$ near 0 and approximate $e^{-0.5}$. Estimate the error and find the smallest interval that you can be sure contains the value.

5. (a) From Taylor's theorem write down an expansion for the remainder when the Taylor

polynomial of degree N for e^x (about $x = 0$) is subtracted from e^x. In what interval does the unknown constant c lie, if $x > 0$?

(b) Show that the remainder has the bounds, if $x > 0$, $\frac{x^{n+1}}{(n+1)!} < R_N < e^x \frac{x^{n+1}}{(n+1)!}$ and use the sandwich rule to show that $R_N \to 0$ as $N \to \infty$. This proves that the Taylor series for e^x does converge to e^x, for any $x > 0$.

3. Fourier series

1. Find the Fourier series for the function $f : [0, 2\pi] \to \mathbb{R}$ given by $f(x) = x^2$.