A topological space is a set X with a specification of the open sets.

A closed set in X is a subset E of X such that E^c is open.

A close point to E is a point $x \in X$ such that if N is a neighborhood of x then $N \cap E \neq \phi$.

A subset E of X is compact if every open cover of E has a finite subcover. i.e.

if $\mathcal{S}$ is a collection of open sets of X and $\bigcup_{U \in \mathcal{S}} U \supseteq E$ then there exists $n \in \mathbb{Z}_{>0}$ ~~such~~ and $U_1, \ldots, U_n \in \mathcal{S}$ such that $U_1 \cup \cdots \cup U_n \supseteq E$.

Theorem (a) If $X = \mathbb{R}$ and $E \subseteq X$ then E is compact if and only if E is closed and bounded.

(b) Let X be a metric space and $E \subseteq X$. If E is compact then E is closed and bounded.

Let $S \subseteq X$. A <u>cluster point</u> of S is an element $q \in X$ such that if N is a neighborhood of p then there exists $s \in S$ such that $s \neq p$ and $s \in N$.

<u>Theorem</u> Let X be a metric space and let $E \subseteq X$. Then E is compact if and only if E satisfies:

if $S \subseteq E$ and S is infinite then there exists $e \in E$ such that e is a cluster point of S.

<u>Main Theorem</u>

(a) Let X and Y be topological spaces and let $f: X \to Y$ be a continuous function.

(aa) If X is connected then $f(X)$ is connected.

(ab) If X is compact then $f(X)$ is compact.

(b) If $f: [a,b] \to \mathbb{R}$ is continuous then

(ba) there exists $c \in [a,b]$ such that

if $x \in [a,b]$ then $f(x) \leq f(c)$.

(bb) there exists $d \in [a,b]$ such that

if $x \in [a,b]$ then $f(x) \geq f(d)$

Mean value theorems

(a) Let $f:[a,b]\to\mathbb{R}$ be continuous and let $c\in(a,b)$ such that if $x\in[a,b]$ then $f(x)<f(c)$. If $f'(c)$ exists then $f'(c)=0$.

(b) Same as (a) except for minimums.

(c) If $f:[a,b]\to\mathbb{R}$ is continuous and f is differentiable on (a,b) then there exists $c\in(a,b)$ such that $f'(c)=0$.

(d) If $f:[a,b]\to\mathbb{R}$ is continuous and f is differentiable on (a,b) then there exists $c\in[a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

(e) If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous and f and g are differentiable on (a,b) then there exists $c\in(a,b)$ such that

$$(f(b)-f(a))g'(c)=(g(b)-g(a))f'(c).$$

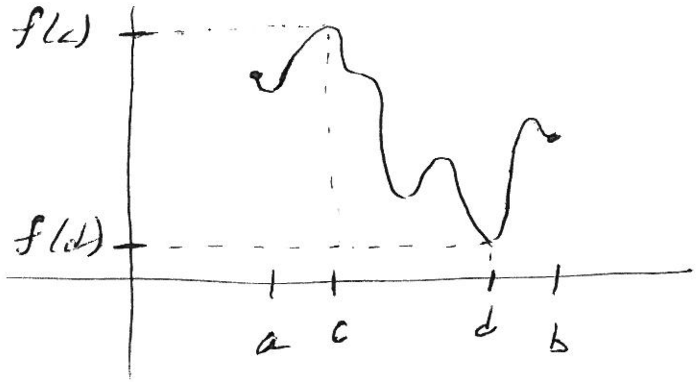
Example Let $f : [0, 2\pi] \to \mathbb{C}$ be given by

$$f(x) = \cos x + i \sin x.$$

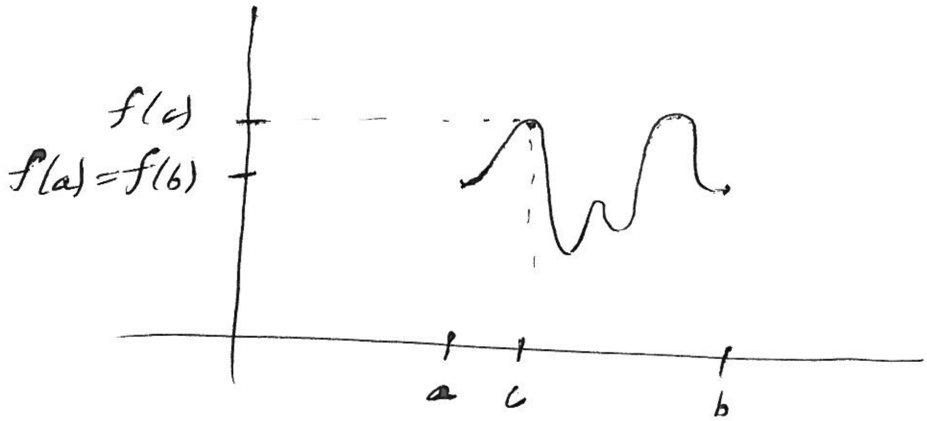
Then $f(0) = f(2\pi)$ but $f'(x)$ is never 0.

Pictures

(a)



(b)



(c)