620-295 Real Analysis with applications Lect. 13, 24.08.2009 (1)

A metric space is a set X with a function

$d: X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $p \in X$ then $d(p,p) = 0$,

(b) If $p, q \in X$ and $p \neq q$ then $d(p,q) \neq 0$,

(c) If $p, q \in X$ then $d(p,q) = d(q,p)$,

(d) If $p, q, r \in X$ then $d(p,q) \leq d(p,r) + d(r,q)$.

Examples (a) $X = \mathbb{R}$ (or $\mathbb{C}$, or $\mathbb{Q}$, or $\mathbb{Z}$, or $\mathbb{Z}_{\geq 0}$, or $\mathbb{Z}_{>0}$)

with $d(x,y) = |y - x|$.

(b) $X = \mathbb{R}^n$ with $d(x,y) = |y - x|$

A sequence in X is a function
$$\begin{aligned} \mathbb{Z}_{>0} &\to X \\ n &\mapsto a_n \end{aligned}$$

The sequence (a_n) converges to L if it gets closer and closer to L.

Write
$$\lim_{n \to \infty} a_n = L, \text{ if } (a_n) \text{ converges to } L.$$

The sequence (a_n) diverges if there does not exist $L \in X$ with $\lim_{n \to \infty} a_n = L$.

<u>Examples</u> (a) $X=\mathbb{R}$, $(a_n)=(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\dots)$

Then $\lim_{n\to\infty} a_n = 0$.

(b) $X=\mathbb{R}$, $(a_n)=\{1,-1,1,-1,1,-1,1,-1,\dots\}$

Then (a_n) diverges.

(c) $X=\mathbb{R}$, $(a_n)=\{1,2,3,4,5,\dots\}$

Then (a_n) diverges.

(d) Pick $x\in\mathbb{R}$ and let $a_n=x^n$ so that

$(a_n)=\{x^1,x^2,x^3,x^4,x^5,x^6,\dots\}$.

If $x=2$, then (a_n) diverges.

If $x=\frac{1}{2}$, then $\lim_{n\to\infty} a_n=0$.

If $x=1$, then $\lim_{n\to\infty} a_n = 1$

If $x=-1$, then (a_n) diverges.

A sequence (a_n) is <u>bounded</u> if the set

$\{d(a_1,0), d(a_2,0),\dots\}$ is bounded above in $\mathbb{R}_{\geq 0}$.

Special definitions for sequences of real numbers

Let (a_n) be a sequence in $\mathbb{R}$.

The sequence (a_n) is increasing if it satisfies

if $n \in \mathbb{Z}_{>0}$ then $a_{n+1} \geq a_n$.

The sequence (a_n) is decreasing if it satisfies

if $n \in \mathbb{Z}_{>0}$ then $a_{n+1} \leq a_n$.

The sequence (a_n) is monotone if it is increasing or decreasing.

The supremum of (a_n) is $\sup\{a_1, a_2, a_3, \ldots\}$

The infimum of (a_n) is

$$\inf a_n = \inf\{a_1, a_2, a_3, a_4, \ldots\}$$

The upper limit and the lower limit of (a_n) are

$$\limsup a_n = \lim_{k\to\infty} b_k \quad \text{and} \quad \liminf a_n = \lim_{k\to\infty} l_k$$

where

$$b_k = \sup\{a_k, a_{k+1}, \ldots\} \quad \text{and} \quad l_k = \inf\{a_k, a_{k+1}, \ldots\}$$

Example $X = \mathbb{R}$ and $a_n = (-1)^n\left(1+\frac{1}{n}\right)$.

So $(a_n) = \{-2, 1\frac{1}{2}, -1\frac{1}{3}, 1\frac{1}{4}, -1\frac{1}{5}, \ldots\}$

and $\sup a_n = 1+\frac{1}{2}$, $\inf a_n = -2$

and $\limsup a_n = 1$ and $\liminf a_n = -1$.

Careful definition of limit of a sequence

Let X be a metric space and let (a_n) be a sequence in X. Let $L \in X$.

The sequence (a_n) converges to L if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, L) < \varepsilon$.

In English: (a_n) converges to L if it passes the following test:

(A) You give me a tolerance ε (say $\varepsilon = .00001$)

(B) I tell you a time to wait N

After time N the sequence is always within ε of L.

Let $\varepsilon \in \mathbb{R}_{>0}$, $L \in X$.

The ε-ball at L is $B_\varepsilon(L) = \{x \in X \mid d(x, L) < \varepsilon\}$.

The sequence (a_n) converges to L if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that $\{a_{N+1}, a_{N+2}, a_{N+3}, \ldots\} \subseteq B_\varepsilon(L)$.

Cauchy sequences

Let X be a metric space and (a_n) a sequence in X.

The sequence (a_n) is a Cauchy sequence if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Proposition Let (a_n) be a convergent sequence. Then (a_n) is a Cauchy sequence.

Proof Let $L \in X$ such that $\lim_{n \to \infty} a_n = L$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

We know there exists $T \in \mathbb{Z}_{>0}$ such that

if $m > T$ then $d(a_m, L) < \varepsilon/2$

if $n > T$ then $d(a_n, L) < \varepsilon/2$

So $d(a_m, a_n) \le d(a_m, L) + d(a_n, L) = \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$.

So (a_n) is Cauchy. //