620-295 Real Analysis with applications

Problem Sheet 2

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1. Groups and Monoids

1. Let S be a set with an associative operation with identity. Show that the identity is unique. (This tells us that any commutative monoid has only one heart.)

2. Let S be a set with an associative operation with identity. Let $s \in S$ and assume that s has an inverse in S. Show that the inverse of s is unique. (This tell us that any element of an abelian group has only one mate.)

3. Let S be a set with identity. Let $s \in S$ and assume that s has an inverse in S. Show that the inverse of the inverse of s is equal to s. (This tells us that $-(-s) = s$.)

4. Let S be an abelian group. Show that if $a + c = b + c$ then $a = b$.

5. Let S be a ring. Show that if $s \in S$ then $s \cdot 0 = 0$.

2. The number systems $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

1. Prove that $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$.

2. Prove that $\sum_{k=1}^{n} \left(2k - 1\right) = n^2$.

3. Prove that $\sum_{k=1}^{n} \left(3k - 2\right) = \frac{1}{2} n(3n - 1)$.

4. Prove that $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.

5. Prove that $\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2(n+1)^2$.

6. Prove that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

7. Prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

8. Define $a_1 = 0, a_{2k} = \frac{1}{2} a_{2k-1}$ and $a_{2k+1} = \frac{1}{2} + a_{2k}$. Show that $a_{2k} = \frac{1}{2} - \left(\frac{1}{2}\right)^{\mathrm{k}}$.

9. Prove that if $n \in \mathbb{Z}_{>0}$ then 3 is a factor of $n^3 - n + 3$.

10. Prove that if $n \in \mathbb{Z}_{>0}$ then 9 is a factor of $10^{n+1} + 3 \cdot 10^n + 5$.

11. Prove that if $n \in \mathbb{Z}_{>0}$ then 4 is a factor of $5^n - 1$.

12. Prove that if $n \in \mathbb{Z}_{>0}$ then $x - y$ is a factor of $x^n - y^n$.

13. Give an example of $s \in \mathbb{Q}$ which has more than one representation as a fraction.

14. Show that $\sqrt{2} \notin \mathbb{Q}$.

15. Show that $\sqrt{3} \notin \mathbb{Q}$.

16. Show that $\sqrt{15} \notin \mathbb{Q}$.

17. Show that $2^{1/3} \notin \mathbb{Q}$.

18. Show that $11^{1/4} \notin \mathbb{Q}$.

19. Show that $16^{1/5} \notin \mathbb{Q}$.

20. Show that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

21. Give an example of $s \in \mathbb{R}$ which has more than one decimal expansion.

22. Compute the decimal expansion of π to 30 digits.

23. Compute the decimal expansion of 2π to 30 digits.

24. Compute the decimal expansion of π^2 to 30 digits.

25. Compute the decimal expansion of $-\pi$ to 30 digits.

26. Compute the decimal expansion of π^{-1} to 30 digits.

27. Show that .9999... = 1.00000... .

28. Compute the decimal expansion of $\sqrt{2}$ to 30 digits.

29. Let $z = x + yi$ with $x,\ y \in \mathbb{R}$. Show that $z^{-1} = \frac{1}{|z|^2}(x - yi)$.

3. Orders

1. Define the following and give an example for each:
   (a) partial order,
   (b) total order,
   (c) order,
   (d) ordered set,
   (e) maximum,
   (f) minimum,
   (g) upper bound,
   (h) lower bound,
   (i) bounded above,
   (j) bounded below,
   (k) least upper bound,
   (l) greatest lower bound,
   (m) supremum,
   (n) infimum,
   (o) intervals.

2. An ordered set S has the *least upper bound property* if it satisfies:
   If $E \subseteq S$, $E \neq \emptyset$, and E is bounded above then $\sup(E)$ exists in S.

3. An ordered set S is *well ordered* if it satisfies:
   If $E \subseteq S$ then E has a minimal element.

4. An ordered set S is *totally ordered* if it satisfies:

If $x, y \in S$ then $x < y$ or $x < y$.

5. An ordered set S is *a lattice* if it satisfies:
   If $x, y \in S$ then $\sup\{x, y\}$ and $\inf\{x, y\}$ exist.

6. Show that $\mathbb{Q}$ does not have the least upper bound property.

7. Show that $\mathbb{R}$ has the least upper bound property.

8. Which of $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{C}$ have the least upper bound property?

9. Which of $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are well ordered?

10. Which of $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are totally ordered?

11. Which of $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are lattices?

12. Let S be a set. Show that the set of subsets of S is partially ordered by inclusion.

13. Define the following and give examples of each:
    (a) ordered monoid,
    (a) ordered group,
    (a) ordered ring,
    (a) ordered field,

14. Let S be an ordered field. Prove the following:
    (a) If $a \in S$ and $a > 0$ then $-a < 0$.
    (b) If $a \in S$ and $a > 0$ then $a^{-1} > 0$.
    (c) If $a, b \in S$, $a > 0$ and $b > 0$ then $ab > 0$.

15. Let S be an ordered group and let $x \in G$. Define the *absolute value* of x.

4. Orders on $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

1. Define the order $\geq$ on $\mathbb{Z}_{>0}$.

2. Define the order $\geq$ on $\mathbb{Z}_{\geq 0}$.

3. Define the order $\geq$ on $\mathbb{Z}$.

4. Define the order $\geq$ on $\mathbb{Q}$.

5. Show that $\frac{a}{b} \leq \frac{c}{d}$ if and only if $abd^2 \leq cdb^2$.

6. Define the order $\geq$ on $\mathbb{R}$.

7. Show that there is no order $\geq$ on $\mathbb{C}$ such that $\mathbb{C}$ is a totally ordered field.

8. Show that if $x, y, z \in \mathbb{R}$ and $x \leq y$ and $y \leq z$ then $x \leq z$.

9. Show that if $x, y \in \mathbb{R}$ and $x \leq y$ and $y \leq x$ then $x = y$.

10. Show that if $x, y, z \in \mathbb{R}$ and $x \leq y$ then $x + z \leq y + z$.

11. Show that if $x, y \in \mathbb{R}$ and $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

12. Show that if $x \in \mathbb{R} - \{0\}$ then $x^2 > 0$.

13. Show that if $x, y \in \mathbb{R}$ and $0 < x < y$ then $y^{-1} < x^{-1}$.

14. (The Archimedean property of $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $nx > y$.

15. Show that the Archimedean property is equivalent to $\mathbb{Z}_{>0}$ is an unbounded subset of $\mathbb{R}$.

16. ($\mathbb{Q}$ is dense $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

17. ($\mathbb{R} - \mathbb{Q}$ is dense $\mathbb{R}$) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{R} - \mathbb{Q}$ such that $x < p < y$.

18. If $x, y \in \mathbb{R}$ and $x < y$ show that there exist infinitely many rational numbers between x and y as well as infinitely many irrational numbers.

19. Let $x \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$. Then there exists a unique $y \in \mathbb{R}_{>0}$ such that $y^n = x$.

20. Find the minimal $N \in \mathbb{Z}_{>0}$ such that $n < 2^n$ for all $n \geq N$.

21. Find the minimal $N \in \mathbb{Z}_{>0}$ such that $n! > 2^n$ for all $n \geq N$.

22. Find the minimal $N \in \mathbb{Z}_{>0}$ such that $2^n > 2n^3$ for all $n \geq N$.

23. For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:

(a) $A = \{p \in \mathbb{Q} | p^2 < 2\}$,
(b) $B = \{p \in \mathbb{Q} | p^2 > 2\}$,
(c) $E_1 = \{r \in \mathbb{Q} | r < 0\}$,
(d) $E_2 = \{r \in \mathbb{Q} | r \leq 0\}$,
(e) $E = \{\frac{1}{n} | n \in \mathbb{Z}_{>0}\}$,
(f) $[0, 1)$,
(g) $\mathbb{Z}_{>0}$,
(h) $\{x \in \mathbb{Q} | x \leq 0 \text{ or } (x > 0 \text{ and } x^2 > 2)\}$,
(i) $\mathbb{Z}$,
(j) $[\sqrt{2}, 2]$,
(k) $(\sqrt{2}, 2)$,
(l) $\{x \in \mathbb{R} | x = \frac{(-1)^n}{n}, \quad n \in \mathbb{Z}_{>0}\}$,
(m) $\left\{\frac{1}{(|n|+1)^2} \middle| n \in \mathbb{Z}\right\}$,
(n) $\left\{n + \frac{1}{n} \middle| n \in \mathbb{Z}_{>0}\right\}$,
(o) $\{2^{-m} - 3^n | m, n \in \mathbb{Z}_{\geq 0}\}$,
(p) $\{x \in \mathbb{R} | x^3 - 4x < 0\}$,
(q) $\{1 + x^2 | x \in \mathbb{R}\}$,

24. Let S be a nonempty subset of $\mathbb{R}$. Show that $x = \sup S$ if and only if
    (a) x is an upper bound of S, and
    (b) for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $y \in S$ such that $x - \varepsilon < y \leq x$.

25. State and prove a characterization of $\inf S$ analogous to the characterization of $\sup S$ in the previous problem.

26. Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that if S is bounded then $c + S = \{c + s \mid s \in \mathbb{R}\}$ is bounded.

27. Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that if S is bounded then $cS = \{cs \mid s \in \mathbb{R}\}$ is bounded.

28. Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that $\sup(c + S) = c + \sup S$.

29. Let $c \in \mathbb{R}_{\geq 0}$ and let S be a subset of $\mathbb{R}$. Show that $\sup(cS) = c \sup S$.

30. Let $c \in \mathbb{R}$ and let S be a subset of $\mathbb{R}$. Show that $\inf(c + S) = c + \inf S$.

31. Let $c \in \mathbb{R}_{\leq 0}$ and let S be a subset of $\mathbb{R}$. Show that $\inf(cS) = c \inf S$.

5. Absolute value

1. Let $x \in \mathbb{R}$. Define $|x|$.

2. Let $x \in \mathbb{C}$. Define $|x|$.

3. Let $x \in \mathbb{R}$. Show that $|x| = |x + 0i|$.

4. Let $x \in \mathbb{R}$. Show that $|-x| = |x|$.

5. Let $x,\ y \in \mathbb{R}$. Show that $|x + y| \leq |x| + |y|$.

6. Let $x,\ y \in \mathbb{C}$. Show that $|x + y| \leq |x| + |y|$.

7. Let $x,\ y,\ z \in \mathbb{R}$. Show that $|x + y + z| \leq |x| + |y| + |z|$.

8. Let $x,\ y,\ z \in \mathbb{C}$. Show that $|x + y + z| \leq |x| + |y| + |z|$.

9. Let $x,\ y \in \mathbb{C}$. Show that $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$.

10. Let $x,\ y \in \mathbb{C}$. Show that $|x + y|^2 = |x|^2 + |y|^2 + 2\mathrm{Re}(a\bar{b})$.

11. Let $x,\ y \in \mathbb{R}$. Show that $|x + y| \geq |\ |x| - |y|\ |$.

12. Let $x,\ y \in \mathbb{R}$. Show that $|x - y| \geq |\ |x| - |y|\ |$.

13. Let $x,\ y,\ z \in \mathbb{R}$. Show that $|x + y + z| \geq |\ |x| - |y| - |z|\ |$.

14. Give solutions to the following inequalities in terms of intervals:
    (a) $|x| > 3$.
    (b) $|1 + 2x| \leq 4$.
    (c) $|x + 2| \geq 5$.
    (d) $|x - 5| < |x + 1|$.
    (e) $|x - 2| < 3$ or $|x + 1| < 1$.
    (f) $|x - 2| < 3$ and $|x + 1| < 1$.

15. Let $a,\ b \in \mathbb{R}$ and let $0 < \varepsilon < |b|$. Show that $\left|\dfrac{a + \varepsilon}{b + \varepsilon}\right| \leq \dfrac{|a| + \varepsilon}{|b| + \varepsilon}$.

16. Prove that if $a_1,\ a_2,\ \ldots,\ a_n \in \mathbb{R}$ then $\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} \left|a_k\right|$.

17. Prove that if $a_1, a_2, \ldots, a_n \in \mathbb{R}$ then $\left|\sum_{k=1}^{n} a_k\right| \geq |a_p| - \sum_{k=1, k \neq p}^{n} \left|a_k\right|$.

6. Inequalities

1. (Bernoulli's inequality) Prove that if $a \in \mathbb{R}$ and $a >$ -1 then $(1+a)^n \geq 1 + na$ for $n \in \mathbb{Z}_{>0}$.

2. Prove that if $x \in \mathbb{R}$ then $1 + x \leq e^x$.

3. Prove that if $x \in \mathbb{R}_{>0}$ then $\log x \geq \frac{x-1}{x}$.

4. Prove that if $x, y \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{R}$ with $0 < p < 1$ then $(x+y)^p \leq x^p + y^p$.

5. (Jensen's inequality) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a convex function. If x_1, $\ldots, x_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in [0, 1]$ with $t_1 + \cdots + t_n = 1$, then $f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1)$ $+ \cdots + t_n f(x_n)$.
6. If $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$ and $t_1, \ldots, t_n \in \mathbb{R}_{\geq 0}$ with $t_1 + \cdots + t_n = 1$, then $t_1 x_1 + \cdots + t_n x_n \geq$ ${x_1}^{t_1} \cdots {x_n}^{t_n}$.