620-295 Real Analysis with applications Lect. ~~7~~; 10/08/2009 ①

4 rules for the derivative $\frac{d}{dx}: \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$.

(a) $\frac{d}{dx}(a(x)+b(x)) = \frac{d}{dx}a(x) + \frac{d}{dx}b(x)$

(b) If $c \in \mathbb{Q}$ and $a(x) \in \mathbb{Q}[[x]]$ then

$$\frac{d}{dx}(c\,a(x)) = c\frac{d}{dx}(a(x))$$

(c) $\frac{d}{dx}x = 1$,

(d) If $a(x), b(x) \in \mathbb{Q}[[x]]$ then

$$\frac{d}{dx}(a(x)b(x)) = a(x)\frac{d}{dx}(b(x)) + \frac{d\,a(x)}{dx}b(x).$$

Proposition If $n \in \mathbb{Z}_{>0}$ then $\frac{dx^n}{dx} = nx^{n-1}$.

Proof Proof by induction.

Base cases: $n=1$. To show: $\frac{dx}{dx} = 1x^0$.

Since $1x^0 = 1$, this follows from (c).

$n=2$: To show $\frac{dx^2}{dx} = 2x$.

$$\frac{dx^2}{dx} = \frac{d}{dx}(x\cdot x) = x\frac{dx}{dx} + \frac{dx}{dx}x = x\cdot 1 + 1\cdot x = 2x.$$

$n=3$: To show $\frac{dx^3}{dx} = 3x^2$

$$\frac{dx^3}{dx} = \frac{d\, x^2 \cdot x}{dx} = x^2 \frac{dx}{dx} + 2x \cdot x = x^2 + 2x^2 = 3x^2.$$

Induction step: Assume $\frac{dx^k}{dx} = k x^{k-1}$, for $k<n$.

To show: $\frac{dx^n}{dx} = n x^{n-1}$.

$$\frac{dx^n}{dx} = \frac{d\, x^{n-1} \cdot x}{dx} = x^{n-1} \frac{dx}{dx} + \frac{d\, x^{n-1}}{dx} \cdot x$$

$$= x^{n-1} \cdot 1 + (n-1) x^{n-2} \cdot x = x^{n-1} + (n-1) x^{n-1}$$

$$= n x^{n-1}. \quad //.$$

Proposition

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n (n+1)(2n+1), \text{ for } n \in \mathbb{Z}_{>0}.$$

Proof Proof by induction.

Base cases: $n=1$. To show: $1^2 = \frac{1}{6} \cdot 1 \cdot (1+1)(2 \cdot 1 + 1)$

$$RHS = \frac{1}{6} \cdot 1 \cdot (1+1)(2 \cdot 1 + 1) = \frac{1}{6} 2 \cdot 3 = 1.$$

$n=2$ To show: $1^2 + 2^2 = \frac{1}{6} \cdot 2(2+1)(2 \cdot 2 + 1)$.

$$LHS = 1^2 + 2^2 = 1 + 4 = 5.$$

$RHS = \frac{1}{6}\, 2(2+1)(2\cdot 2+1) = \frac{1}{6}\, 2\cdot 3\cdot 5 = 5.$

<u>$n=3$</u> To show: $1^2+2^2+3^2 = \frac{1}{6}\, 3\cdot(3+1)(2\cdot 3+1)$

$LHS = 1^2+2^2+3^2 = 5+3^2.$

$RHS = \frac{1}{6}\, 3\cdot 4\cdot 7 = \frac{1}{6}(2+1)(3+1)(5+2)$

$= \frac{1}{6}\, 2\cdot 3\cdot 5 + \frac{1}{6}\, 1\cdot 4\cdot 7 + \frac{1}{6}\, 2(1\cdot 5+3\cdot 2+1\cdot 2)$

<u>Induction step:</u>

Assume $\sum_{k=1}^{r} k^2 = \frac{1}{6} r(r+1)(2r+1)$, for $r<n$.

To show: $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+(n-1)^2+n^2 = n^2+\sum_{k=1}^{n-1} k^2$$

$= n^2+\frac{1}{6}(n-1)(n-1+1)(2(n-1)+1)$, by induction,

$= \frac{1}{6}6n^2+\frac{1}{6}(n-1)n(2n-1)$

$= \frac{1}{6}n(6n+(n-1)(2n-1))$

$= \frac{1}{6}\cancel{8}n(6n+2n^2-n-2n+1)$

$= \frac{1}{6}n(2n^2+3n+1) = \frac{1}{6}n(n+1)(2n+1).$

Taylor's Theorem

Let $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots \quad \in \mathbb{Q}[[x]]$.

$$\frac{da}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$\frac{d^2a}{dx^2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3\, a_4 x^2 + 5\cdot 4 a_5 x^3 + \cdots$$

$$\frac{d^3a}{dx^3} = 3\cdot 2a_3 + 4\cdot 3\cdot 2\, a_4 x + 5\cdot 4\cdot 3\, a_5 x^2 + 6\cdot 5\cdot 4 a_6 x^3 + \cdots$$

$$\frac{d^4a}{dx^4} = 4\cdot 3\cdot 2\, a_4 + 5\cdot 4\cdot 3\cdot 2\, a_5 x + 6\cdot 5\cdot 4\cdot 3\, a_6 x^2 + \cdots$$

In general,

$$\frac{d^k a}{dx^k} = k!\, a_k + (k+1)!\, a_{k+1} x + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(k+r)!}{r!} a_{k+r} x^r .$$

So

$$\left.\frac{d^k a}{dx^k}\right|_{x=0} = k!\, a_k + 0 + 0 + 0 + \cdots = k!\, a_k .$$

So $a_k = \dfrac{1}{k!}\left(\left.\dfrac{d^k a}{dx^k}\right|_{x=0} \right)$.

Examples

(1) If you know

$$e^0 = 1, \quad \frac{d e^x}{dx} = e^x, \quad \frac{d^2 e^x}{dx} = e^x, \ \ldots$$

then you know

$$\frac{1}{k!}\left(\frac{d^k e^x}{dx}\Bigg|_{x=0}\right) = \frac{1}{k!} e^x\Big|_{x=0} = \frac{1}{k!}$$

and

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

(2) If you know

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \frac{d \sin x}{dx} = \cos x$$

and $\dfrac{d \cos x}{dx} = -\sin x$,

then you know that if $k \in \mathbb{Z}_{>0}$ then

$$\frac{d^k \sin x}{dx^k} = \begin{cases} (-1)^{k/2} \sin x, & \text{if } k \text{ is even} \\ (-1)^{\lfloor k/2 \rfloor} \cos x, & \text{if } k \text{ is odd} \end{cases}$$

since

$$\frac{d^2 \sin x}{dx^2} = -\sin x, \quad \frac{d^3 \sin x}{dx^3} = -\cos x, \quad \frac{d^4 \sin x}{dx^4} = \sin x$$

So

$$\frac{1}{k!}\frac{d^k \sin x}{dx^k}\Bigg|_{x=0} = \begin{cases} \frac{1}{k!}\cdot 0, & \text{if } k \text{ is even} \\ \frac{1}{k!}(-1)^{\lfloor k/2 \rfloor}, & \text{if } k \text{ is odd,} \end{cases}$$

and $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$