Sheet 1

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville VIC 3010 Australia
aram@unimelb.edu.au

and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

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1. Numbers

1. Define the following sets and give examples of elements of each:
   (a) the set of positive integers,
   (b) the set of nonnegative integers,
   (c) the set of integers,
   (d) the set of rational numbers,
   (e) the set of real numbers,
   (f) the set of complex numbers,
   (g) the set of algebraic numbers.

2. Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$.
   (a) Define $\frac{a}{b} + \frac{c}{d}$ and $\frac{a}{b}\frac{c}{d}$.
   (b) Show that $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.
   (c) Show that if $\frac{a}{b} + \frac{c}{d} = \frac{c}{d}$ then $\frac{a}{b} = \frac{0}{1}$.
   (d) Show that if $\frac{a}{b} + \frac{c}{d} = \frac{0}{7}$ then $\frac{c}{d} = \frac{\text{-a}}{b}$.

3. Compute the decimal expansion of $\frac{3651}{342}$.

4. State and prove the Pythagorean Theorem.

5. Show that $\sqrt{2} \notin \mathbb{Q}$.

6. Graph $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, and $\overline{\mathbb{Q}}$, as subsets of $\mathbb{C}$.

7. State the fundamental theorem of algebra.

8. Compute and graph the following:
   (a) $\dfrac{-15+i}{4+2i}$,
   (b) $(3-2i)^3$,
   (c) $\sqrt{2i}$,
   (d) $\left(27^{1/3}\right)^4$,
   (e) $27^{(4+1/3)}$.

9. Compute and graph the following:
   (a) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$,
   (b) $(1+i)^n + (1-i)^n$, for $n \in \mathbb{Z}_{\geq 0}$.

10. Let $z = x + iy$ with $x,\ y \in \mathbb{R}$. Compute and graph the following:
   (a) $\dfrac{1}{z}$,
   (b) z^4,
   (c) $\left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$.

11. Show that the conjugate of $\dfrac{z}{z^2+1}$ is equal to $\dfrac{\bar{z}}{\bar{z}^2+1}$.

12. Define the following and give examples:
   (a) set,
   (b) subset,
   (c) equal sets,
   (d) union,
   (e) intersection,
   (f) product of sets,
   (g) emptyset,
   (h) function,
   (i) well defined function,
   (j) equal functions,
   (k) injective,
   (l) surjective,
   (m) bijective.

13. Explain why $\sqrt{x}$ is not a function.

14. Define the following:
   (a) composition of functions,
   (b) identity map on S,

(c) inverse function,

(d) $\sqrt{x}$,

(e) $x^{1/7}$,

(f) $\log(x)$,

(g) $\sin^{-1} x$,

(h) $\tan^{-1} x$,

(i) $\cosh^{-1} x$,

15. Define the following and give examples:
    (a) monoid without identity,
    (b) monoid,
    (c) group,
    (d) commutative monoid,
    (e) abelian group,
    (f) ring,
    (g) commutative ring,
    (h) field,
    (i) division ring.

16. Define the following and give examples:
    (a) operation,
    (b) commutative,
    (c) associative.

17. Give an example of an operation that is not commutative and not associative.

18. Give an example of an operation that is associative but not commutative.

19. Define the following sets and give examples of elements of each:
    (a) $\mathbb{Q}[x]$,
    (b) $\mathbb{Q}[[x]]$,
    (c) $\mathbb{Q}(x)$,
    (d) $\mathbb{Q}((x))$.

20. Let $D : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$ be a function such that

    (D1) If $f, g \in \mathbb{Q}[x]$ then $D(f+g) = D(f) + D(g)$,

    (D2) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$ then $D(cf) = cD(f)$,

    (D3) If $f, g \in \mathbb{Q}[x]$ then $D(fg) = fD(g) + D(f)g$ and

    (D4) $D(x) = 1$.

    (a) Compute $D(x^n)$, for $n \in \mathbb{Z}_{\geq 0}$.

    (b) Let $f = c_0 + c_1 + c_2 + c_3 + c_4 + \ldots$. Show that $c_k = \frac{1}{k!}(D^k f)|_{x=0}$.

21. Write the following as elements of $\mathbb{Q}[x]$:

(a) $\frac{1-x^n}{1-x}$,

(b) e^x

(c) $\sin x$

(d) $\sin(1+x)$,

(e) $\cos x$

(f) $\frac{1}{1-x}$,

(g) $(1+x)^7$,

(h) $(1+x)^{1/7}$.