28.05.2025
Calculus Lect.
A.Ram

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Flies in $\mathbb{R}^2$

Position: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Velocity: $\vec{v} = \vec{r}\,'(t) = \frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$

Acceleration: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$

Speed: $s = \|\vec{v}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$s^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

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$$\frac{ds^2}{dt} = 2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}$$

$$= 2\left\langle\left(\frac{dx}{dt}, \frac{dy}{dt}\right), \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)\right\rangle = 2\langle v, a\rangle$$

$$= 2(\vec{v}\cdot\vec{a})$$

Then s is increasing exactly when s^2 is increasing which is exactly when $\frac{ds^2}{dt} > 0$ which is exactly when $\vec{v}\cdot\vec{a} > 0$.

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$$\int_2^3 \frac{1}{x\sqrt{x^2-1}}\,dx = \int_{x=2}^{x=3} \frac{1}{x\sqrt{x^2-1}}\,dx$$

Let $x = \sec(\theta)$

$$\frac{dx}{d\theta} = \tan(\theta)\sec(\theta)$$

then

$$\int_{x=2}^{x=3} \frac{1}{\sec(\theta)\sqrt{\sec(\theta)^2-1}} \frac{dx}{d\theta}\,d\theta$$

$$= \int_{x=2}^{x=3} \frac{1}{\sec(\theta)\sqrt{\tan(\theta)^2}} \tan(\theta)\sec(\theta)\,d\theta$$

$$= \int_{x=2}^{x=3} d\theta = \theta\Big]_{x=2}^{x=3}$$

$$= \theta\Big]_{\sec(\theta)=2}^{\sec(\theta)=3} = \theta\Big]_{\frac{1}{\cos(\theta)}=2}^{\frac{1}{\cos(\theta)}=3}$$

$$= \theta\Big]_{\cos(\theta)=\frac{1}{2}}^{\cos(\theta)=\frac{1}{3}} = \theta\Big]_{\theta=\arccos(\frac{1}{2})}^{\theta=\arccos(\frac{1}{3})}$$

$$= \arccos(\tfrac{1}{3}) - \arccos(\tfrac{1}{2})$$

$$\sin(\theta)^2 + \cos(\theta)^2 = 1$$

$$\frac{\sin(\theta)^2}{\cos(\theta)^2} + \frac{\cos(\theta)^2}{\cos(\theta)^2} = \frac{1}{\cos(\theta)^2}$$

$$\tan(\theta)^2 + 1 = \sec(\theta)^2.$$

So $\sec(\theta)^2 - 1 = \tan(\theta)^2.$

$$\frac{d\sec(\theta)}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{\cos(\theta)}\right)$$

$$= \frac{d}{d\theta}\left(\cos(\theta)^{-1}\right)$$

$$= (-1)\cos(\theta)^{-2} \cdot \frac{d}{d\theta}(\cos(\theta))$$

$$= \frac{-1}{\cos(\theta)^2}(-\sin(\theta))$$

$$= \frac{\sin(\theta)}{\cos(\theta)\cos(\theta)}$$

$$= \tan(\theta)\sec(\theta)$$

$$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_{x=\sqrt{2}}^{x=2} \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

Let $x = \sec(\theta)$

$$\frac{dx}{d\theta} = \tan(\theta)\sec(\theta)$$

Then

$$\int_{x=\sqrt{2}}^{x=2} \frac{1}{\sec(\theta)^2\sqrt{\sec(\theta)^2-1}}\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_{x=\sqrt{2}}^{x=2} \frac{1}{\sec(\theta)^2\sqrt{\tan(\theta)^2}}\tan(\theta)\sec(\theta)\,d\theta$$

$$= \int_{x=\sqrt{2}}^{x=2} \frac{1}{\sec(\theta)}\,d\theta = \int_{x=\sqrt{2}}^{x=2} \cos(\theta)\,d\theta = \sin(\theta)\Big]_{x=\sqrt{2}}^{x=2}$$

$$= \sin(\theta)\Big]_{\sec(\theta)=\sqrt{2}}^{\sec(\theta)=2} = \sin(\theta)\Big]_{\cos(\theta)=\frac{1}{\sqrt{2}}}^{\cos(\theta)=\frac{1}{2}}$$

$$= \sqrt{1-\cos(\theta)^2}\Big]_{\cos(\theta)=\frac{1}{\sqrt{2}}}^{\cos(\theta)=\frac{1}{2}}$$

$$= \sqrt{1-\left(\frac{1}{2}\right)^2} - \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{4}} - \sqrt{\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3}-\sqrt{2}}{2}.$$

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$$\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin(2\theta)^2\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{(e^{i2\theta}-e^{-i2\theta})}{2i}\right)^2 d\theta = \frac{-1}{4}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left((e^{i2\theta})^2 - 2e^{i2\theta}e^{-i2\theta} + (e^{-i2\theta})^2\right) d\theta$$

$$= \frac{-1}{4}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (e^{i4\theta}+e^{-i4\theta}-2)\,d\theta = \frac{-1}{4}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (2\cos(4\theta)-2)\,d\theta$$

$$= \frac{-1}{4}\left(2\frac{\sin(4\theta)}{4} - 2\theta\right)\Big]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \left(-\frac{\sin(4\theta)}{8} + \frac{\theta}{2}\right)\Big]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \left(-\frac{\sin(2\pi)}{8} + \frac{\pi}{4}\right) - \left(\frac{-\sin(0)}{8} + \frac{0}{2}\right)$$

$$= -0 + \frac{\pi}{4} + 0 - 0 = \frac{\pi}{4}$$