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Calculus Lect
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# Velocity, acceleration, speed

Suppose the position of the fly is

$$r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} \quad \text{for } t \in \mathbb{R}.$$

The velocity is

$$\vec{v} = r'(t) = \frac{dr}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}.$$

The speed is

$$s = \|r'(t)\| = \|\vec{v}\|$$

The acceleration is

$$\vec{a} = r''(t) = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} = \frac{d\vec{v}}{dt}.$$

7.33 (a) $r(t) = (t^2+1)\hat{\imath} + (\frac{1}{3}t^3+1)\hat{\jmath}$

$$v = r'(t) = 2t\,\hat{\imath} + t^2\hat{\jmath}$$

$$a = r''(t) = 2\hat{\imath} + 2t\hat{\jmath}$$

$$s = \|v\| = \sqrt{(2t)^2 + (t^2)^2} = \sqrt{4t^2+t^4}.$$

When is the speed 0? When $t=0$.

When is $s$ increasing? When $s^2 = 4t^2+t^4$ is increasing. i.e. when $\frac{ds^2}{dt} = 8t + 4t^3 > 0$.

So $s$ is increasing when $4t(2+t^2) > 0$.

So $s$ is increasing when $t>0$.

(b) $r(t) = \hat{i} + 3t^3\hat{j}$

$v = r'(t) = 9t^2\hat{j}$

$a = r''(t) = 18t\hat{j}$

$s = \|v\| = \sqrt{(9t^2)^2} = 9t^2$

When is speed 0? When $t=0$.

When is speed increasing?

When $\frac{ds}{dt} > 0$ so that $18t > 0$.

So $s$ is increasing when $t > 0$.

(c) $r(t) = 2t\hat{i} - (3t^3+1)\hat{j}$

$v = r'(t) = 2\hat{i} - 9t^2\hat{j}$

$a = r''(t) = -18t\hat{j}$

$s = \|v\| = \sqrt{2^2 + (-9t^2)^2} = \sqrt{4 + 81t^4}$

When is speed 0? Never

When is speed increasing?

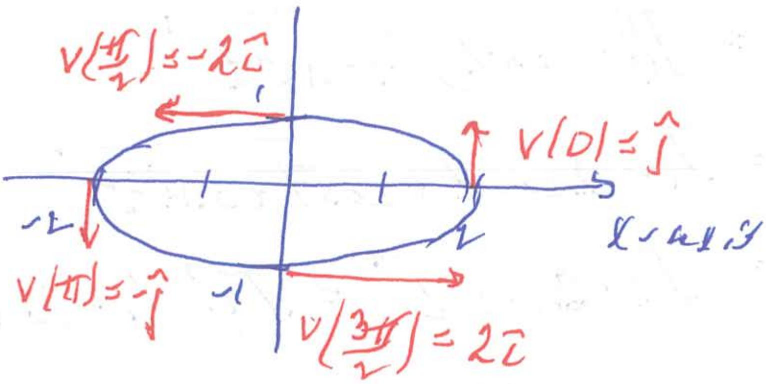
When $s^2$ is increasing. $s^2 = 4 + 81t^4$.

When $\frac{ds^2}{dt} > 0$. When $4 \cdot 81t^3 > 0$.

So speed is increasing when $t > 0$.

(d) $r(t) = 2\cos(t)\hat{i} + \sin(t)\hat{j}$

$$v = r'(t) = -2\sin(t)\hat{i} + \cos(t)\hat{j}$$




$$a = r''(t) = -2\cos(t)\hat{i} - \sin(t)\hat{j}$$

$$s = \|r'(t)\| = \sqrt{4\cos(t)^2 + \sin(t)^2}$$

$$= \sqrt{3\cos(t)^2 + 1}$$

$$s^2 = 3\cos(t)^2 + 1$$

$$\frac{ds^2}{dt} = -6\cos(t)\sin(t) = -3\sin(2t)$$

When is speed 0? Never.

When is speed increasing? When $\frac{ds^2}{dt} > 0$.

$-3\sin(2t) > 0$ when $t \in (2k\pi, 2k\pi + \frac{\pi}{2})$ for $k \in \mathbb{Z}$

and $t \in ((2k+1)\pi, (2k+1) + \frac{\pi}{2})$ for $k \in \mathbb{Z}$.

(e) $r(t) = (t - \sin(t))\hat{\imath} + (1 - \cos(t))\hat{\jmath}$

$v = r'(t) = (1 - \cos(t))\hat{\imath} + \sin(t)\hat{\jmath}$

$a = r''(t) = \sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$

$s = \|r'(t)\| = \sqrt{(1-\cos(t))^2 + \sin(t)^2}$

$= \sqrt{1 - 2\cos(t) + \cos(t)^2 + \sin(t)^2}$

$= \sqrt{2 - 2\cos(t)}$

When is speed 0? When $t \in \{2k\pi \mid k \in \mathbb{Z}\}$.

When is speed increasing?

When $s^2$ is increasing. $s^2 = 2 - 2\cos(t)$

When $\frac{ds^2}{dt} = 2\sin(t) > 0$

So speed is increasing when

$$t \in \bigcup_{k \in \mathbb{Z}} (2k\pi, (2k+1)\pi).$$